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Understanding Rational Numbers[1]: Are the Textbooks helping?

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Introduction

Classroom discussions with students of pre-service elementary education programme (B El ED[2]) on the concept of rational numbers has made me reflect on their understanding of rational numbers. During one of the discussions with third-year B El ED students, I asked them what is a rational number? The students gave me the definition of rational number that they had learnt in school. They said, "A rational number is any number in the form a/b, where b≠0", while "an irrational number is a number that cannot be expressed in the form a/b, where b≠0".

The students were then asked, if 1/3 is a rational number? They answered in the affirmative. They were, then, asked where will you place 10/3? Is it a rational or an irrational number? The students answered, "It is a rational number!" Next, they were asked where will you place 3.3333? Is it a rational or an irrational number? The students answered: "It is an irrational number". Then, the following was deduced from the above interaction

If 10/3 = rational number, and 3.3333... = irrational number

Then, 10/3 = 3.3333... = irrational number, and 10/3 = rational number.

Therefore, all rational numbers = all irrational numbers.

That is when the students realised that there was something wrong in the

1. Students in elementary school may be more familiar with the term 'fractions' as compared to 'rational numbers'. However, in this article, the term rational number is used as it captivates the meaning of the term fraction and many more concepts in it. The rationale of this will become clearer to readers when the theoretical framework is discussed.
2. B EL ED is a four-year integrated teacher education programme, which was started by the University of Delhi in 1994. The course continues to prepare reflective pedagogues in elementary school.

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deductions that were made. I reflected as to why the students were not able to make connections? Such experiences were not specific to a particular batch of students. This understanding of rational numbers is reflected across batches. The questions that came to my mind were — why have not these students made a connection between the two representations of rational number? If the students are unable to recognise the same number in the representations, then have they understood the concept of rational number? Why is it that even after engaging with the concept of rational numbers since elementary school, the students are not able to move beyond the definition of rational number?

The answers to the questions raised lies in the experiences that the students had gained in school while engaging with rational numbers. The classroom experiences of the students could be in tandem with the experiences explicated in textbooks. Just studying the textbooks may not present a complete picture of classroom engagements of the students. However, they may help develop an understanding of the content that is likely to be discussed in the classroom.

Derek Stolp (2010), while discussing the dependence of a teacher on textbooks, shares a teacher's comments on textbook: "The textbook provides a map and if we (teachers) follow it, we won't get lost. The more faithfully we follow this map, we believe the better mathematicians we will become."

This reflects the teacher's dependence on textbooks, and thus, it is imperative to understand the content of the textbooks to see whether they are helping the students acquire an understanding of rational numbers beyond the definitions and procedures. The textbooks will be analysed to comprehend how they unfold the concept of rational numbers. As per the National Curriculum Framework–2005, the students are to engage with rational numbers[3] from Class IV to VIII. This study chooses to focus on the initial years of the students' learning of rational numbers. Thus, NCERT mathematics textbooks of Class IV and V were studied to understand the experiences being provided to the students.

What are Fractions and Rational Numbers?

Fraction is a term that is used in primary classes to address part-whole relations of rational numbers. Peter Gould (2005) observes that the greatest advantage of fraction or part-whole models is that it is a readily available option for teachers and the text to introduce the symbolic notation to students. Thus, fractions form an integral part of primary school mathematics.

3. The students, as per the National Curriculum Framework (NCF)–2005, are introduced to fractions from Class IV onwards.

The mathematics textbook of Class VII published by the NCERT (2007) defines 'rational number' as a number that can be expressed in the form p/q, where p and q are integers and $q \neq 0$. This basic definition of rational number that any number in the form p/q, where $q \neq 0$ allows to classify mathematically the concepts of fraction, decimal, measure, ratio and proportion, and percentage under rational numbers, while maintaining the defining characteristics of each of the concepts. The theoretical framework for this collation of concepts under rational number was provided by Thomas Kieren. Kieren (1978) in his research on rational numbers established that the concept of rational number is not a single construct, and in 1980, he established the five ideas of rational numbers, which he called sub-constructs that could form the basis for understanding rational numbers. They were—fractions or part-whole sub-construct, quotient sub-construct, measure sub-construct, ratio sub-construct and operator sub-construct. The details of the theory is explicated under the heading 'The Sub-construct Theory of Rational Number'.

Rational numbers include all natural numbers and integers as depicted in *Fig. 7.1*. Real numbers are larger sets, which contain all rational numbers and numbers, which cannot be represented in the form of p/q, $q \neq 0$ (irrational numbers).

The students are introduced to the concept of fractions or part-whole sub-construct of rational numbers in Class IV (NCF, 2005). However, the term rational number is not introduced to them. The term rational number explicitly features in the curriculum only in Class VII (NCF–2005). The textbook beyond elementary school does not have rational numbers as an explicit chapter. It is assumed that students beyond elementary school are proficient in this concept. Through the elementary years (Classes IV to VIII), the textbooks expose students to halves and quarters, parts and whole, fraction, ratio and proportion, decimal, and comparing quantities.

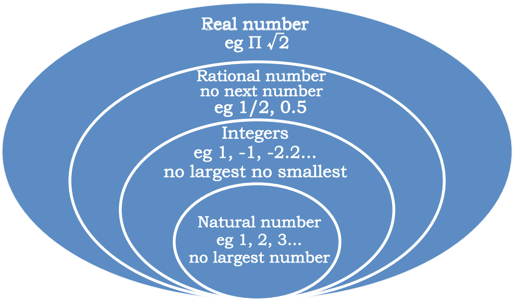

Fig. 7.1: A simplified model of the extension of number domains from natural to real numbers and changes in the level of abstraction with each enlargement (Merenluoto, et al., 2007)

The Sub-construct Theory of Rational Number

According to the framework by Kieren (1976), the Sub-construct Theory of Rational Number looks at rational number as a synthesis of five concepts. This is the uniqueness of this theory as it allows a comprehensive perspective of rational number. Thus, to understand the concept of rational numbers, the students must engage with each sub-construct. The focus at this point should be to comprehend what each sub-construct signifies.

Fraction or part-whole sub-construct: Part-whole sub-construct assigns a number of equal parts of a unit (unit can be a continuous quantity or a set of discrete objects) out of the total number of equal parts into which it is divided.

Example: Identifying 3/4 in a whole, which has been partitioned into eights (Charalambous, et al., 2007)

Quotient sub-construct: Partitioning or dividing into equal parts is the basis for a rational number to be understood as a quotient. In particular, the rational number x/y indicates the numerical value obtained when 'x' is divided by 'y', where 'x' and 'y' are whole numbers (S. Lamon, 1999).

Example: Three pizzas are evenly divided among four children. How many pizza pieces will each child get? (Behr, et al., 1993)

Measure sub-construct: Measurement interpretation is pointed towards the use of number line as a physical model. The reason for the term 'measurement interpretation' is that rational numbers are defined as measures — the unit of measure is the distance on the number line from 0 to 1 (Kieren, 1976).

Example: In a given line segment from 0 to 1, identify where 1/3 would lie.

Ratio sub-construct: The ratio interpretation of rational numbers supports the notion of comparison between two quantities. Thus, it is considered comparative index rather than a number (Charalombos, 2007).

Example: Three boys share one pizza and seven girls share three. Who gets more pizza pieces, a girl or a boy? (Lamon, 1999)

Operator sub-construct: In the operator interpretation of rational numbers, rational numbers are regarded as functions applied to some number, object, or set. For example, 3/4 is thought of as a function applied to some number. The significant relationship is the comparison between the quantities that is acted upon. The operator defines the relationship quantity out or quantity in (Lamon, 1999).

Example: Sahil has 3/4 as many toy cars as I have. I have 40 toy cars. How many does Sahil have?

Challenges to the Learning of Rational Numbers

Moss and Case (1999) argue that the domain of rational numbers is considered to be one of the most complex mathematical concepts in elementary school because understanding rational numbers requires a conceptual shift, i.e., numbers must be understood in multiplicative relations. Since the concept of rational numbers is embedded in the multiplicative conceptual field, it has a distinct character as compared to the whole number concept and its operations (Vergnaud, 1983). Hence, many of the successful strategies, which the students may have developed for whole numbers, will not work for rational numbers, like the largest number is the longest number.

Another example, is '2 is less than 3' however '1/2 will not be less than 1/3' on the contrary '1/2 is more than 1/3'. Researchers (Moss, 2005; Ni and Zhou, 2005) have identified this to be one of the major causes of difficulty for students. They named such errors arising out of applying natural number knowledge in situations when it is not appropriate as natural number knowledge interference. Another challenge for students is that they are taught while learning the properties of natural numbers that there is no natural number between any two consecutive numbers. But in case of rational numbers, this property is not true. On the contrary, there are infinite number of rational numbers between any two consecutive natural numbers (Lamon, 2012; Stafylidou and Vosniadou, 2010).

The students face challenge in the representation of rational numbers because of multiplicity in the representation of a rational number — diagrammatically and symbolically. When rational numbers are to be represented diagrammatically, there are three options — set model, area model and linear model. According to Jeremy Kilpatrick, et al. (2001), the task for students while trying to understand rational numbers is to recognise these distinctions, and at the same time, construct relations among them to generate a coherent concept of rational numbers. And, when a rational number is represented symbolically, there are many options, like p/q notation, decimal representation, percentages, and equivalent fractions, for example, the number 1/2 can be represented as 0.5, 50 per cent, 0.50, 0.500, 2/4, 3/6, 4/8, etc. The list is endless. This multiplicity in representation, symbolically and diagrammatically, offers many challenges to the students and all are not able to overcome it.

The question that needs to be answered is — are the students given enough opportunities to engage with the concept of rational numbers to develop a comprehensive understanding of the concept. The textbook must provide space and opportunities to students to challenge the misconceptions they develop from 'natural number knowledge interference' and multiplicity in the representation of the same rational number.

Textbooks and Rational Number Sub-constructs

When textbooks were scrutinised vis-à-vis the sub-construct theory of rational numbers, it was observed that only three of the five sub-constructs were represented namely — fraction or part-whole sub-construct, measure sub-construct and operator sub-construct. Ratio and quotient sub-construct were missing from primary class textbooks. However, unlike other sub-constructs, ratio sub-construct has separate chapters across textbooks for middle school. Among the sub-constructs represented in the textbooks of Classes IV and V, it was observed that fractions or part-whole sub-construct was given the most exposure.

Researchers (Seibert and Gaskin, 2006; and Van de Walle, 2013) inform that the actions of partitioning (3/4 is three one-fourths, where 1/4 is the result of dividing the whole into four parts, and 1/4 is one of the four parts), iterating (3/4 is three one-fourths, where 1/4 is the amount such that four of the one-forth join together to make a whole), and sharing (sharing of four apples between two persons) have been recognised as important to the understanding of part-whole relations.

The textbooks have several examples of partitioning task (a situation where a whole is divided into parts) but those of sharing task (a situation where a whole or more is shared among two or more people) are few (one example of sharing *halwa* is on page 63, Math Magic, Textbook in Mathematics for Class V, 2008, NCERT). Iteration task (counting using a part of a fraction to arrive at a whole) is represented in the set model in both the textbooks by using currency, and weights and measures as contexts. Also, there is one question in the area model (page 63, Math Magic, Textbook in Mathematics for Class V, 2008, NCERT), wherein the students have to find the 'whole from a part'. Instances of completing the picture are also introduced in the textbook of Class IV. But are the students able to connect the act of completing the picture to iteration?

Fig. 7.2: An example of iteration by completing the figure

An important aspect of the textbook is that it encourages students to think that there are multiple ways to arrive at a solution, like in the case of dividing a figure into four parts. The students are asked to divide the figure in multiple ways. They are explicitly asked to draw lines along which the figure can be divided. Space is also provided for the students to explore several options. Thus, the students are made to think of multiple ways of dividing a figure, thereby, forcing them to explore different ways to divide the same.

There are examples of all three models of the diagrammatic representation — set model (all tasks involve countable number of objects), area model (all tasks involve sharing something that can be cut into pieces), and linear model (all tasks involve number lines, these are therefore,

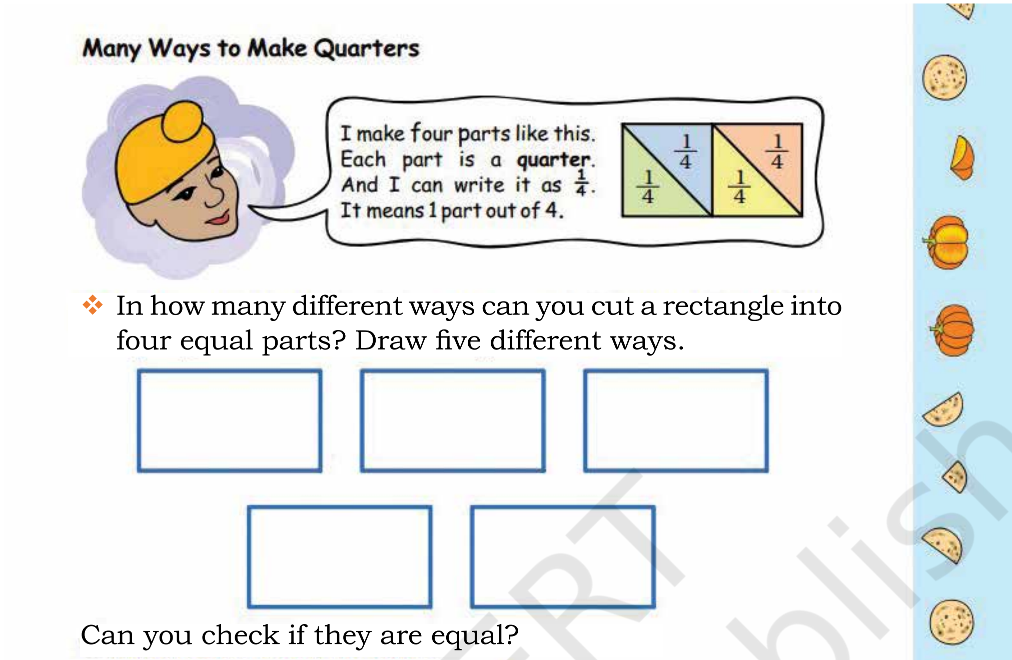

Fig. 7.3: An example from Class IV

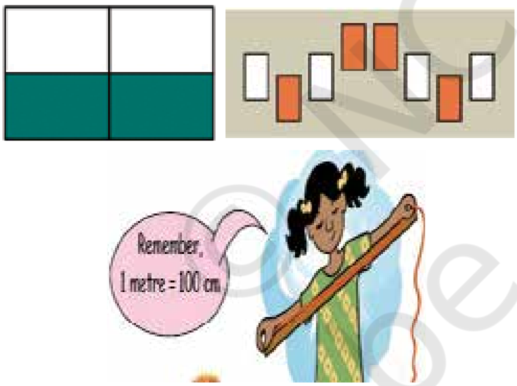

Fig. 7.4: Examples of the three models of representation

tasks of measure sub-construct) — in the textbooks.

Given above are examples on area model (page 101, Math Magic, Textbook in Mathematics for Class IV, 2007, NCERT), set model (page 102), and linear model (page 103). The textbook, however, tends to focus more on the area model, and there is relatively less space for tasks of set model. There is a little space for tasks in the linear model or measure sub-construct. There is only one instance in the Mathematics textbook of Class IV of a child using a measuring tape, which qualifies to be a task on measure sub-construct. Thus, the only engagement that the students have with measure sub-construct is in the textbook of Class IV, and that too, just one activity.

Operator sub-construct does have a presence in the textbooks. There is a story of Birbal (page 55) and a task on Ramu's vegetable field (page 58) in the Mathematics textbook of Class V. Through these examples, the students get an idea of operator sub-construct and these will help them understand the concept of multiplication of rational numbers.

Like ratio sub-construct, there was no instance of quotient sub-construct in these textbooks. However, for ratios, there is an entire chapter in the textbook of Class VI. But for quotient sub-construct, there is only a small section on the division of fraction, which has been addressed mechanically in the textbook for Class IV. Examples like '4/5 can also mean the share of one person when five persons are sharing four *poories*', can be given to students. Such tasks can be introduced diagrammatically in the primary classes itself. Should the students wait that long to get introduced to examples like these for quotient sub-construct? If examples like these are introduced in earlier classes by encouraging the children to represent the portions diagrammatically, while dealing with small numbers, they are likely to learn about the division of fractions much earlier than in Class VII. An option available for the textbook is to include sharing tasks under fraction sub-construct and link them to the ideas of quotient sub-construct.

Suggestions

A textbook must provide students with information about all five sub-constructs. Jeremy Kilpatrick, et al. (2001) in the book, *Adding it up,* remark that students will be able to generate a coherent understanding of the concept only if they are able to "recognise these (five sub-constructs) distinctions and at the same time, construct relations among them". The textbook (Class V, NCERT) has not addressed all sub-constructs of the rational number sub-construct theory. And in places where the sub-constructs were included, the tasks were few.

Thomas E. Hodges, et al. (2008) in their article titled 'Fraction Representation: The not-so-common Denominator among Textbook' comment that since the textbook lacks multiple representations, set model, area model and linear model to support students' learning, "therefore, it is up to the teachers to encourage students to use and reflect on their use of representations" (page 82). However, the textbook should have examples of all types of representation rather than putting pressure on teachers to find tasks to complete the work done in the textbook on all aspects of rational number. Van de Walle (2013) in the book titled *Teaching student-centered Mathematics* (page 295) observes: "As a teacher, you will not know that whether they really understand the meaning of a fraction, such as 1/4 unless you have seen a student represent one-fourth using area, length and set models."

A textbook must have space for students to engage with issues that emerge from natural number knowledge interference. Opportunities must be provided to them to challenge their understanding of the concept of rational numbers with the understanding they have of natural numbers. If these misconceptions are not challenged in the early years of engaging with rational numbers, they may get crystallised rather than the idea of rational numbers.

Conclusion

The above analysis reveals that the textbooks (Class IV and V, NCERT) is far from providing a holistic understanding of rational numbers. The ratio and the quotient sub-constructs are not represented in NCERT textbooks. The textbook must incorporate examples and tasks for ratio sub-construct, quotient sub-construct, and have more tasks for measure sub-construct. Even though ratio sub-construct and measure sub-construct are addressed in the middle school, the students must be to introduced to these concepts in the primary classes itself.

The ideas that have been described above are an attempt to help teachers understand the concept of rational numbers holistically. The teaching of rational numbers must be in tandem with the sub-construct theory of rational numbers and the textbook must support teachers in this endeavour by including experiences needed to understand rational numbers in the textbook. Tasks of varied nature are necessary to provide an opportunity to students to explore and understand each sub-construct of rational numbers.

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