Teaching of Mathematics*

Abstract

As a part of the development of the National Curriculum Framework–2005*, twenty one National Focus Groups (NFGs) were constituted to reflect upon diverse themes drawn from the area of school education. Each NFG brought out a research-based position paper. For our readers, we present here the text of the position paper on "Teaching of Mathematics". While enlisting the main goal of mathematics education in schools on the mathematisation of the child's thinking, the paper proposes a shift from mathematics content to mathematics learning environment offering a multiplicity of approaches, procedures and solutions. Such learning environment helps in removing fear of mathematics from children's minds and is crucial for liberating school mathematics from the tyranny of the one right answer. The vision of excellent mathematical education, as recommended by the position paper, is based on the twin premises that all students can learn mathematics and that students need to learn mathematics.*

1. Goals of Mathematics Education

What are the main goals of mathematics education in schools? Simply stated, there is one main goal— the mathematisation of the child's thought processes. In the words of David Wheeler, it is "more useful to know how to mathematise than to know a lot of mathematics".

According to George Polya, we can think of two kinds of aims for school education: a good and narrow aim, which of turning out employable adults who (eventually) contribute to social and economic development; and a higher aim, that of developing the inner resources of the growing child2. With regard to school mathematics, the former aim specifically relates to numeracy. Primary schools teach numbers and operations on them, measurement of quantities, fractions, percentages and ratios, all these are important for numeracy.

What about the higher aim? In developing a child's inner resources, the role that mathematics plays is mostly about thinking. Clarity of thought and pursuing assumptions to logical conclusions is central to the mathematical enterprise. There are many ways of thinking, and the kind of thinking one learns in mathematics is an ability to handle abstractions.

Even more importantly, what mathematics offers is a way of doing

* *National Curriculum Framework-2005*, Position Paper, National Focus Group on Teaching of Mathematics, NCERT, 2006

things: to be able to solve mathematical problems, and more generally, to have the right attitude for problem solving and to be able to attack all kinds of problems in a systematic manner.

This calls for a curriculum that is ambitious, coherent and teaches important mathematics. It should be ambitious in the sense that it seeks to achieve the higher aim mentioned above, rather than (only) the narrower aim. It should be coherent in the sense that the variety of methods and skills available piecemeal (in arithmetic, algebra, geometry) cohere into an ability to address problems that come from science and social studies in high school. It should be important in the sense that students feel the need to solve such problems, that teachers and students find it worth their time and energy addressing these problems and those mathematicians consider it an activity that is mathematically worthwhile. Note that such importance is not a given thing, and curriculum can help shape it. An important consequence of such requirements is that school mathematics must be activity-oriented.

In the Indian context, there is a centrality of concern which has an impact on all areas of school education, namely that of universalisation of schooling. This has two important implications for the discussion on curriculum, especially mathematics. Firstly, schooling is a legal right, and mathematics being a compulsory subject of study, access to quality mathematics education is every child's right. Keeping in mind the Indian reality, where few children have access to expensive material, we want mathematics education that is affordable to every child, and at the same time, enjoyable. This implies that the mathematics taught is situated in the child's lived reality, and that for the system, it is not the subject that matters more than the child, but the other way about.

Secondly, in a country where nearly half the children drop out of school during the elementary stage, mathematics curricula cannot be grounded only on preparation for higher secondary and university education. Even if we achieve our targeted universalisation goals, during the next decade, we will still have a substantial proportion of children exiting the system after Class VIII. It is then fair to ask what eight years of school mathematics offers for such children in terms of the challenges they will face afterwards.

Much has been written about *life skills* and linkage of school education to livelihood. It is certainly true that most of the skills taught at the primary stage are useful in everyday life. However, a reorientation of the curriculum towards addressing the 'higher aims' mentioned above, will make better use of the time children spend in schools in terms of the problem solving and analytical skills it builds in children, and prepare them better to encounter a wide variety of problems in life.

Our reflections on the place of mathematics teaching in the curricular framework are positioned on these twin concerns: what mathematics education can do to engage the mind of every student, and how it can strengthen the student's resources. We describe our vision of mathematics in school, attempt to delineate the core areas of concern, and offer recommendations that address the concerns, based on these twin perspectives.

Many of our considerations in what follows have been shaped by discussions of Mathematics Curriculum in NCTM, USA[3], and the New Jersey Mathematics Coalition[4], the Mathematics academic content standards of the California State Board of Education[5], the Singapore Mathematics Curriculum[6], the Mathematics Learning Area statements of Australia and New Zealand[7], and the national curricula of France, Hungary[8] and the United Kingdom[9]. Ferrini-Mundi et al (eds.) offer an interesting discussion comparing national curriculum and teaching practice in mathematics in France with that of Brazil, Egypt, Japan, Kenya, Sweden and the USA10.

2. A Vision Statement

In our vision, school mathematics takes place in a situation where:

- Children learn to *enjoy mathematics*: This is an important goal, based on the premise that mathematics can be both used and enjoyed life-long, and hence that school is best placed to create such a taste for mathematics. On the other hand, creating (or not removing) a fear of mathematics can deprive children of an important faculty for life.
- Children learn *important mathematics*: Equating mathematics with formulas and mechanical procedures does great harm. Understanding when and how a mathematical technique is to be used is always more important than recalling the technique from memory (which may easily be done using a book), and the school needs to create such understanding.
- Children see mathematics as something to talk about, to communicate, to discuss among themselves, to work together on. Making mathematics *a part of children's life experience* is the best mathematics education possible.
- Children pose and solve *meaningful problems*: In school, mathematics is the domain which formally addresses problem solving as a skill. Considering that this is an ability of use in all of one's life, techniques and approaches learnt in school have great value. Mathematics also provides an opportunity to make up interesting problems, and create new dialogues thereby.
- Children use abstractions to perceive relationships, to see structure, to reason about things, to argue the truth or falsity of statements. *Logical thinking* is a great gift mathematics can offer us, and inculcating such habits of thought and communication in children is a principal goal of teaching mathematics.
- Children understand the basic *structure of mathematics*: Arithmetic, algebra, geometry and trigonometry, the basic content areas of school mathematics, all offer a methodology for abstraction, structuration and generalisation. Appreciating the scope and power of mathematics refines our instincts in a unique manner.
- Teachers expect to engage every child in class: Settling for anything less can only act towards systematic

exclusion, in the long run. Adequately challenging the talented even while ensuring the participation of all children is a challenge, and offering teachers means and resources to do this is essential for the health of the system.

Such a vision is based on a diagnosis of what we consider to be the central problems afflicting school mathematics education in the country today, as also on what we perceive can be done, and ought to be done.

Before we present the vision, a quick look at the history of mathematics curricular framework is in order.

3. A Brief History

Etymologically, the term 'curriculum' which has been derived from the Latin root means 'race course'. The word race is suggestive of time and course - the path. Obviously, curriculum was seen as the prescribed course of study to be covered in a prescribed time frame. But, evolution of curriculum as a field of study began in 1890's only, albeit of the fact that thinkers of education were interested in exploring the field for centuries. Johann Friedrich Herbart (1776-1841), a German thinker, is generally associated with the evolution of curriculum-field. Herbart had emphasised the importance of 'selection' and 'organisation' of content in his theories of teaching/learning. The first book devoted to the theme of curriculum entitled, The Curriculum was published in 1918 by Franklin Bobbitt followed by another book How to make Curriculum in 1924. In 1926, the National society for the study of education in America published the year book devoted to the theme of curriculum-The Foundation and Technique of Curriculum Construction. This way the curriculum development movement, from its beginning in 1890s, started becoming a vigorous educational movement across the world.

School systems are a relatively new phenomenon in historical terms, having developed only during the past two hundred years or so. Before then, there existed schools in parts of the West, as an appendage to religious organisations. The purpose of these schools was to produce an educated cleric. Interest in mathematics was rudimentary- 'the different kinds of numbers and the various shapes and sufficient astronomy to help to determine the dates of religious rituals'. However, in India the practice of education was a well established phenomenon**.** Arithmetic and astronomy were core components of the course of study. Astronomy was considered essential for determining auspicious times for performing religious rituals and sacrifices. Geometry was taught because it was required for the construction of sacrificial altars and 'havan kunds' of various shapes and sizes. With the arrival of the British, the system of education underwent a major change. Western system of education was introduced to educate Indians on western lines for the smooth functioning of the Empire.

However, much of the curriculum development in mathematics has taken place during the past thirty/forty years. This is because of the new technological revolution which has an impact on society as great as the industrial revolution. Modern technology is, therefore, causing, and will increasingly cause educational aims to be rethought, making curriculum development a dynamic process. To a scanning eye, mathematics itself is being directly affected by the modern technology as new branches are developed in response to new technological needs, leaving some 'time-hallowed' techniques redundant. In addition, teaching of mathematics also gets affected in order to keep pace with new developments in technology. Moreover, there exists a strong similarity of mathematics syllabi all over the world, with the result that any change which comes from the curriculum developers elsewhere is often copied or tried by others. India, for example, got swayed with the wave of new mathematics. Later, following the trends in other countries, new mathematics also receded here. To conclude, the various trends in curriculum development we observe no longer remain a static process, but a dynamic one. Its focus from 'selection' and 'organisation' of the informational material shifts to the development of a curriculum that 'manifests life in its reality'.

In 1937, when Gandhiji propounded the idea of basic education, the Zakir Hussain committee was appointed to elaborate on this idea. It recommended: 'Knowledge of mathematics is an essential part of any curriculum. Every child is expected to work out the ordinary calculations required in the course of his craft work or his personal and community concerns and activities.' The Secondary Education Commission appointed in 1952 also emphasised the need for mathematics as a compulsory subject in the schools.

In line with the recommendations of the National Policy on Education, 1968, when the NCERT published its "Curriculum for the Ten Year School", it remarked that the 'advent of automation and cybernatics in this century marks the beginning of the new scientific industrial revolution and makes it all the more imperative to devote special attention to the study of mathematics'. It stressed on an 'investigatory approach' in the teaching of mathematics.

The National Policy on Education 1986 went further

Mathematics should be visualised as the vehicle to train a child to think, reason, analyse and to articulate logically. Apart from being a specific subject, it should be treated as a concomitant to any subject involving analysis and reasoning.

The National Curriculum Framework for School Education (NCFSE) 2000 document echoes such sentiments as well. Yet, despite this history of exhortations, mathematics education has remained pretty much the same, focussed on narrow aims.

4. Problems In Teaching And Learning of Mathematics

Any analysis of mathematics education in our schools will identify a range of issues as problematic. We structure our understanding of these issues around the following four problems which we deem to be the core areas of concern:

1. A sense of fear and failure regarding mathematics among a majority of children,
2. A curriculum that disappoints both a talented minority as well as the

non-participating majority at the same time,

3. Crude methods of assessment that encourage perception of mathematics as mechanical computation, and
4. Lack of teacher preparation and support in the teaching of mathematics.

Each of these can and need to be expanded on, since they concern the curricular framework in essential ways.

4.1 Fear and Failure

If any subject area of study evokes wide emotional comment, it is mathematics. While no one educated in Tamil would profess (or at the least, not without a sense of shame) ignorance of any Tirukkural, it is quite the social norm for anyone to proudly declare that (She) he never could learn mathematics. While these may be adult attitudes, among children (who are compelled to pass mathematics examinations) there is often fear and anxiety. Mathematics anxiety and 'math phobia' are terms that are used in popular literature.

In the Indian context, there is a special dimension to such anxiety. With the universalisation of elementary education made a national priority, and elementary education a legal right, at this historic juncture, a serious attempt must be made to look into every aspect that alienates children in school and contributes towards their non-participation, eventually leading to their dropping out of the system. If any subject taught in school plays a significant role in alienating children and causing them to stop attending school, perhaps mathematics, which inspires so much dread, must take a big part of the blame.

Such fear is closely linked to a sense of failure. By Class III or IV, many children start seeing themselves as unable to cope with the demands made by mathematics. In high school, among children who fail only in one or two subjects in year-end examinations, and hence, are detained, the maximum numbers fail in mathematics. This statistic pursues us right through to Class X, which is when the Indian state issues a certificate of education to a student. The largest numbers of Board Exam failures also happen in mathematics.

There are many perceptive studies and analyses on what causes fear of mathematics in schools. Central among them is the cumulative nature of mathematics. If you struggle with decimals, then you will struggle with percentages; if you struggle with percentages, then you will struggle with algebra and other mathematics subjects as well. The other principal reason is said to be the predominance of symbolic language. When symbols are manipulated without understanding, after a point, boredom and bewilderment dominate many children, and dissociation develops.

Failure in mathematics could be read through social indicators as well. Structural problems in Indian education, reflecting structures of social discrimination, by way of class, caste and gender, contribute further to failure (and perceived failure) in mathematics education as well. Prevalent social attitudes which see girls as incapable of mathematics, or which, for centuries, have associated formal computational abilities with the upper castes, deepen such failure by way of creating self-fulfilling expectations.

A special mention must be made of problems created by the language used in textbooks, especially at the elementary level. For a vast majority of Indian children, the language of mathematics learnt in school is far removed from their everyday speech, and especially forbidding. This becomes a major force of alienation in its own right.

4.2 Disappointing Curriculum

Any mathematics curriculum that emphasises procedure and knowledge of formulas over understanding is bound to enhance anxiety. The prevalent practice of school mathematics goes further: a silent majority gives up early on, remaining content to fail in mathematics, or at best, to see it through, maintaining a minimal level of achievement. For these children, what the curriculum offers is a store of mathematical facts, borrowed temporarily while preparing for tests.

On the other hand, it is widely acknowledged that more than in any other content discipline, mathematics is the subject that also sees great motivation and talent even at an early age in a small number of children12. These are children who take to quantisation and algebra easily, and carry on with great facility.

What the curriculum offers for such children is also intense disappointment. By not offering conceptual depth, by not challenging them, the curriculum settles for minimal use of their motivation. Learning procedures may be easy for them, but their understanding and capacity for reasoning remain under-exercised.

4.3 Crude Assessment

We talked of fear and failure. While what happens in class may alienate, it never evokes panic, as does the examination. Most of the problems cited above relate to the tyranny of procedure and memorization of formulas in school mathematics, and the central reason for the ascendancy of procedure is the nature of assessment and evaluation. Tests are designed (only) for assessing a student's knowledge of procedure and memory of formulas and facts, and given the criticality of examination performance in school life, concept learning is replaced by procedural memory. Those children who cannot do such replacement successfully experience panic, and suffer failure.

While mathematics is the major ground for formal problem solving in school, it is also the only arena where children see little room for play in answering questions. Every question in mathematics is seen to have one unique answer, and either you know it or you don't. In Language, Social Studies, or even in Science, you may try and demonstrate partial knowledge, but (as the students see it), there is no scope for doing so in mathematics. Obviously, such a perception is easily coupled to anxiety.

Amazingly, while there has been a great deal of research in mathematics education and some of it has led to changes in pedagogy and curriculum, the area that has seen little change in our schools over a hundred years or more is evaluation procedures in mathematics. It is not accidental that even a quarterly examination in Class VII is not very different in style from a Board examination in Class X, and the same pattern dominates even the end-of-chapter exercises given in textbooks. It is always application of some piece of information given in the text to solve a specific problem that tests use of formalism. Such antiquated and crude methods of assessment have to be thoroughly overhauled if any basic change is to be brought about.

4.4 Inadequate Teacher Preparation

More so than any other content discipline, mathematics education relies very heavily on the preparation that the teacher has, in her own understanding of mathematics, of the nature of mathematics, and in her bag of pedagogic techniques. Textbook-centred pedagogy dulls the teacher's own mathematics activity.

At two ends of the spectrum, mathematics teaching poses special problems. At the primary level, most teachers assume that they know all the mathematics needed, and in the absence of any specific pedagogic training, simply try and uncritically reproduce the techniques they experienced in their school days. Often this ends up perpetuating problems across time and space.

At the secondary and higher secondary level, some teachers face a different situation. The syllabi have considerably changed since their school days, and in the absence of systematic and continuing education programmes for teachers, their fundamentals in many concept areas are not strong. This encourages reliance on 'notes' available in the market, offering little breadth or depth for the students.

While inadequate teacher preparation and support act negatively on all of school mathematics, at the primary stage, its main consequence is this: mathematics pedagogy rarely resonates with the findings of children's psychology. At the upper primary stage, when the language of abstractions is formalised in algebra, inadequate teacher preparation reflects as inability to link formal mathematics with experiential learning. Later on, it reflects as incapacity to offer connections within mathematics or across subject areas to applications in the sciences, thus depriving students of important motivation and appreciation.

4.5 Other Systemic Problems

We wish to briefly mention a few other systemic sources of problems as well. One major problem is that of **compartmentalisation**: there is very little systematic communication between primary school and high school teachers of mathematics, and none at all between high school and college teachers of mathematics. Most school teachers have never even seen, let alone interacted with or consulted, research mathematicians. Those involved in teacher education are again typically outside the realm of college or research mathematics.

Another important problem is that of curricular acceleration: a generation ago, calculus was first encountered by a student in college. Another generation earlier, analytical geometry was considered college mathematics. But these are all part of school curriculum now. Such acceleration has naturally meant pruning of some topics: there is far less solid geometry or spherical geometry now. One reason for the narrowing is that calculus and differential equations are critically important in undergraduate sciences, technology and engineering, and hence, it is felt that early introduction of these topics helps students proceeding further on these lines. Whatever the logic, the shape of mathematics education has become taller and more spindly, rather than broad and rounded.

While we have mentioned gender as a systemic issue, it is worth understanding the problem in some detail. Mathematics tends to be regarded as a 'masculine domain'. This perception is aided by the complete lack of references in textbooks to women mathematicians, the absence of social concerns in the designing of curricula, which would enable children questioning received gender ideologies, and the absence of reference to women's lives in problems. A study of mathematics textbooks found that in the problem sums, not a single reference was made to women's clothing, although several problems referred to the buying of cloth, etc.[13]

Classroom research also indicates a fairly systematic devaluation of girls as incapable of 'mastering' mathematics, even when they perform reasonably well at verbal as well as cognitive tasks in mathematics. It has been seen that teachers tend to address boys more than girls, which feeds into the construction of the normative mathematics learner as male. Also, when instructional decisions are in teachers' hands, their gendered constructions colour the mathematical learning strategies of girls and boys, with the latter using more invented strategies for problem-solving, which reflects greater conceptual understanding.14 Studies have shown that teachers tend to attribute boys' mathematical 'success' more to ability, and girls' success more to effort.15 Classroom discourses also give some indication of how the 'masculinising' of mathematics occurs, and the profound influence of gender ideologies in patterning notions of academic competence in school.16 With performance in mathematics signifying school 'success', girls are clearly at the losing end.

5. Recommendations

While the litany of problems and challenges magnifies the distance we need to travel to arrive at the vision articulated above, it also offers hope by way of pointing us where we need to go and what steps we may/must take.

We summarise what we believe to be the central directions for action towards our stated vision. We group them again into four central themes:

1. Shifting the focus of mathematics education from achieving 'narrow' goals to 'higher' goals,
2. Engaging every student with a sense of success, while at the same time offering conceptual challenges to the emerging mathematician,
3. Changing modes of assessment to examine students' mathematisation abilities rather than procedural knowledge,
4. Enriching teachers with a variety of mathematical resources.

There is some need for elaboration. How can the advocated shift to 'higher' goals remove fear of mathematics in children? Is it indeed possible to simultaneously address the silent majority and the motivated minority? How indeed can we assess processes rather than knowledge? We briefly address these concerns below.

5.1 Towards the Higher Goals

The shift that we advocate, from 'narrow' goals to 'higher' goals, is best summarised as a shift in focus from mathematical content to mathematical learning environments.

The content areas of mathematics addressed in our schools do offer a solid foundation. While there can be disputes over what gets taught at which grade, and over the level of detail included in a specific theme, there is broad agreement that the content areas (arithmetic, algebra, geometry, mensuration, trigonometry, data analysis) cover essential ground.

What can be levelled as a major criticism against our extant curriculum and pedagogy is its failure with regard to mathematical processes. We mean a whole range of processes here: formal problem solving, use of heuristics, estimation and approximation, optimisation, use of patterns, visualisation, representation, reasoning and proof, making connections, mathematical communication. Giving importance to these processes constitutes the difference between doing mathematics and swallowing mathematics, between mathematisation of thinking and memorising formulas, between trivial mathematics and important mathematics, between working towards the narrow aims and addressing the higher aims.

In school mathematics, certainly emphasis does need to be attached to factual knowledge, procedural fluency and conceptual understanding. New knowledge is to be constructed from experience and prior knowledge using conceptual elements. However, invariably, emphasis on procedure gains ascendancy at the cost of conceptual understanding as well as construction of knowledge based on experience. This can be seen as a central cause for the fear of mathematics in children.

On the other hand, the emphasis on exploratory problem solving, activities and the processes referred to above constitute learning environments that invite participation, engage children, and offer a sense of success. Transforming our classrooms in this manner, and designing mathematics curricula that enable such a transformation, is to be accorded the highest priority.

5.1.1 Processes

It is worth explaining the kind of processes we have referred to and their place in the curricular framework. Admittedly, such processes cut across subject areas, but we wish to insist that they are central to mathematics. This is to be seen in contrast with mathematics being equated to exact but abstruse knowledge with an all-or-nothing character.

Formal problem solving, at least in schools, exists only in the realm of mathematics. But for physics lessons in the secondary stage and after, there are no other situations outside of mathematics where children address themselves to problem solving. Given this, and the fact that this is an important 'life skill' that a school can teach, mathematics education needs to be far more conscious of what tactics it can offer. As it stands, problem solving only amounts to doing exercises that illustrate specific definitions in the text. Worse, textbook problems reduce solutions to knowledge of specific tricks, of no validity outside the lesson where they are located.

On the other hand, many general tactics can indeed be taught, progressively during the stages of school. Techniques like abstraction, quantification, analogy, case analysis, reduction to simpler situations, even guess-and-verify, are useful in many problem contexts. Moreover, when children learn a variety of approaches (over time), their toolkit gets richer and they also learn which approach is best when.

This brings us to the use of heuristics, or rules of thumb. Unfortunately, mathematics is considered to be 'exact' where one uses 'the appropriate formula'. To find a property of some triangle, it is often useful to first investigate the special case when the triangle is right angled, and then look at the general case afterwards. Such heuristics do not always work, but when they do, they give answers to many other problems as well. Examples of heuristics abound when we apply mathematics in the sciences. Most scientists, engineers and mathematicians use a big bag of heuristics–a fact carefully hidden by our school textbooks.

Scientists regard estimation of quantities and approximating solutions, when exact ones are not available, to be absolutely essential skills. The physicist Fermi was famous for posing estimation problems based on everyday life and showing how they helped in nuclear physics. Indeed, when a farmer estimates the yield of a particular crop, considerable skills in estimation and approximation are used. School mathematics can play a significant role in developing and honing such useful skills, and it is a pity that this is almost entirely ignored.

Optimisation is never even recognised as a skill in schools. Yet, when we wish to decide on a set of goods to purchase, spending less than a fixed amount, we optimise Rs. 100 can buy us A and B or C, D and E in different quantities, and we decide. Two different routes can take us to the same destination and each has different advantages or disadvantages. Exact solutions to most optimisation problems are hard, but intelligent choice based on best use of available information is a mathematical skill that can be taught. Often, the numerical or geometrical facility needed is available at the upper primary stage. Developing a series of such situations and abilities can make school mathematics enjoyable as well as directly useful.

Visualisation and representation are again skills unaddressed outside mathematics curriculum, and hence, mathematics needs to develop these far more consciously than is done now. Modelling situations using quantities, shapes and forms is the best use of mathematics. Such representations aid visualisation and reasoning clarify essentials, help us discard irrelevant information. Rather sadly, representations are taught as ends in themselves. For example, equations are taught, but the use of an equation to represent the relationship between force and acceleration is not examined. What we need are illustrations that show a multiplicity of representations so that the relative advantages can be understood. For example, a fraction can be written in the form p/q but can also be visualised as a point on the number line; both representations are useful, and appropriate in different contexts. Learning this about fractions is far more useful than arithmetic of fractions.

This also brings us to the need for making connections, within mathematics, and between mathematics and other subjects of study. Children learn to draw graphs of functional relationships between data, but fail to think of such a graph when encountering equations in physics or chemistry. That, algebra offers a language for succinct substitutable statements in science needs underlining and can serve as motivation for many children. Eugene Wigner once spoke of the unreasonable effectiveness of mathematics in the sciences. Our children need to appreciate the fact that mathematics is an effective instrument in science.

The importance of systematic reasoning in mathematics cannot be overemphasised, and is intimately tied to notions of aesthetics and elegance dear to mathematicians. Proof is important, but equating proof with deduction, as done in schools, does violence to the notion. Sometimes, a picture suffices as a proof, a construction proves a claim rigorously. The social notion of proof as a process that convinces a sceptical adversary is important for the practice of mathematics. Therefore, school mathematics should encourage proof as a systematic way of argumentation. The aim should be to develop arguments, evaluate arguments, make and investigate conjectures, and understand that there are various methods of reasoning.

Another important element of process is mathematical communication. Precise and unambiguous use of language and rigour in formulation are important characteristics of mathematical treatment, and these constitute values to be imparted by way of mathematics education. The use of jargon in mathematics is deliberate, conscious and stylised. Mathematicians discuss what appro-priate notation is, since, good notation is held to aid thought. As children grow older, they should be taught to appreciate the significance of such conventions and their use. For instance, this means that setting up of equations should get as much coverage as solving them.

In discussing many of these skills and processes, we have repeatedly referred to offering a *multiplicity* of approaches, procedures, solutions. We see this as crucial for liberating school mathematics from the tyranny of the one right answer, found by applying the one algorithm taught. When many ways are available, one can compare them, decide which is appropriate when, and in the process gain insight. And such a multiplicity is available for most mathematical contexts, all through school, starting from the primary stage. For instance, when we wish to divide 102 by 8, we could do long division, or try 10 first, then 15, and decide that the answer lies in between and work at narrowing the gap.

It is important to acknowledge that mathematical competence is situated and shaped by the social situations and the activities in which learning occurs. Hence, school mathematics has to be in close relation to the social worlds of children where they are engaged in mathematical activities as a part of daily life. Open-ended problems, involving multiple approaches and not solely based on arriving at a final, unitary, correct answer are important so that an external source of validation (the teacher, textbooks, guidebooks) is not habitually sought for mathematical claims. The unitary approach acts to disadvantage all learners, but often acts to disadvantage girls in particular.

5.1.2 Mathematics that people use

An emphasis on the processes discussed above also enables children to appreciate the relevance of mathematics to people's lives. In Indian villages, it is commonly seen that people who are not formally educated use many modes of mental mathematics. What may be called folk algorithms exist for not only mentally performing number operations, but also for measurement, estimation, understanding of shapes and aesthetics. Appreciating the richness of these methods can enrich the child's perception of mathematics. Many children are immersed in situations where they see and learn the use of these methods, and relating such knowledge to what is formally learnt as mathematics can be inspiring and additionally motivating.

For instance, in Southern India, *kolams* (complex figures drawn on the floor using a white powder, similar to rangoli in the north, but ordinarily without colour) are seen in front of houses. A new *kolam* is created each day and a great variety of them are used. Typically women draw *kolams*, and many even participate in competitions. The grammar of these *kolams*, the classes of closed curves they use, the symmetries that they exploit - these are matters that mathematics education in schools can address, to the great benefit of students. Similarly, art, architecture and music offer intricate examples that help children appreciate the cultural grounding of mathematics.

5.1.3 Use of technology

Technology can greatly aid the process of mathematical exploration, and clever use of such aids can help engage students. Calculators are typically seen as aiding arithmetical operations; while this is true, calculators are of much greater pedagogic value. Indeed, if one asks whether calculators should be permitted in examinations, the answer is that it is quite unnecessary for examiners to raise questions that necessitate the use of calculators. On the contrary, in a non-threatening atmosphere, children can use calculators to study iteration of many algebraic functions. For instance, starting with an arbitrary large number and repeatedly finding the square root to see how soon the sequence converges to 1, is illuminating. Even phenomena like chaos can be easily comprehended with such iterators.

If ordinary calculators can offer such possibilities, the potential of graphing calculators and computers for mathematical exploration is far higher. However, these are expensive, and in a country where the vast majority of children cannot afford more than one notebook, such use is luxurious. It is here that governmental action, to provide appropriate alternative low-cost technology, may be appropriate. Research in this direction will be greatly beneficial to school education.

It must be understood that there is a spectrum of technology use in mathematics education, and calculators or computers are at one end of the spectrum. While notebooks and blackboards are the other end, use of graph paper, geo boards, abacus, geometry boxes etc. is crucial. Innovations in the design and use of such material must be encouraged so that their use makes school mathematics enjoyable and meaningful.

5.2 Mathematics for All

A systemic goal that needs to be underlined and internalised in the entire system is universal inclusion. This means acknowledging that forms of social discrimination work in the context of mathematics education as well and addressing means for redress. For instance, gendered attitudes which consider mathematics to be unimportant for girls, have to be systematically challenged in school. In India, even caste based discrimination manifests in such terms, and the system cannot afford to treat such attitudes by default.

Inclusion is a fundamental principle. Children With Special Needs, especially children with physical and mental disabilities, have as much right as every other child to learn mathematics, and their needs (in terms of pedagogy, learning material etc.) have to be addressed seriously. The conceptual world of mathematics can bring great joy to these children, and it is our responsibility not to deprive them of such education.

One important implication in taking Mathematics for all seriously is that even the language used in our textbooks must be sensitive to language uses of all children. This is critical for primary education, and this may be achievable only by a multiplicity of textbooks.

While the emphasised shift towards learning environments is essential for engaging the currently non-participating majority in our classrooms, it does not in any way mean dilution of standards. We are not advising here that the mathematics class, rather than boring the majority, ends up boring the already motivated minority. On the other hand, a case can be made that such open problem situations offer greater gradations in challenges, and hence, offer more for these few children as well.

It is widely acknowledged that mathematical talent can be detected early, in a way that is not observable in more complex fields such as literature and history. That is, it is possible to present challenging tasks to highly talented youngsters. The history of the task may be ignored; the necessary machinery is minimal; and the manner in which such youngsters express their insights does not require elaboration in order to generate mathematical inquiry.

All this is to say that challenging all children according to their mathematical taste is indeed possible. But this calls for systemic mechanisms, especially in textbooks. In India, few children have access to any mathematical material outside their mathematics textbooks, and hence, structuring textbooks to offer such a variety of content is important.

In addition, we also need to consider mechanisms for identification and nurturing of such talent, especially in rural areas, by means of support outside main school hours. Every district needs at least a few centres accessible to children where such mathematical activity is undertaken periodically. Networking such talent is another way of strengthening it.

5.2.1 Assessment

Given that mathematics is a compulsory subject in all school years, all summative evaluation must take into account the concerns of universalisation. Since the Board examination for Class X is for a certificate given by the State, implications of certified failure must be considered seriously. Given the reality of the educational scenario, the fact that Class X is a terminal point for many is relevant; applying the same single standard of assessment for these students as well as for rendering eligibility for the higher secondary stage seems indefensible. When we legally bind all children to complete ten years of schooling, the SSLC certificate of passing that the State issues should be seen as a basic requirement rather than a certificate of competence or expertise.

Keeping these considerations in mind, and given the high failure rate in mathematics, we suggest that the Board examinations be restructured. They must ensure that all numerate citizens pass and become eligible for a State certificate. (What constitutes numeracy in a citizen may be a matter of social policy.) Nearly half the content of the examination may be geared towards this.

However, the rest of the examination needs to challenge students far more than it does now, emphasising competence and expertise rather than memory. Evaluating conceptual understanding rather than fast computational ability in the Board examinations will send a signal of intent to the entire system, and over a period of time, cause a shift in pedagogy as well.

These remarks pertain to all forms of summative examinations at the school level as well. Multiple modes of assessment, rather than the unique test pattern, need to be encouraged. This calls for a great deal of research and a wide variety of assessment models to be created and widely disseminated.

5.3 Teacher Support

The systemic changes that we have advocated require substantial investments of time, energy and support on the part of teachers. Professional development, affecting the beliefs, attitudes, knowledge and practices of teachers in the school, is central to achieving this change. In order for the vision described in this paper to become a reality, it is critical that professional development focuses on mathematics specifically. Generic 'teacher training' does not provide the understanding of content, of instructional techniques, and of critical issues in mathematics education that is needed by classroom teachers.

There are many mechanisms that need to be ensured to offer better teacher support and professional development, but the essential and central requirement is that of a large treasury of resource material which teachers can access freely as well as contribute to. Further, networking of teachers so that expertise and experience can be shared is important. In addition, identifying and nurturing resource teachers can greatly help the process. Regional mathematics libraries may be built to act as resource centres.

An important area of concern is the teacher's own perception of what mathematics is, and what constitutes the goals of mathematics education. Many of the processes we have outlined above are not considered to be central by most mathematics teachers, mainly because of the way they were taught, and a lack of any later training on such processes.

Offering a range of material to teachers that enriches their understanding of the subject, provides insights into the conceptual and historical development of the subject and helps them innovate in their classrooms is the best means of teacher support. For this, providing channels of communication with college teachers and research mathematicians will be of great help. When teachers network among themselves and link up with teachers in universities, their pedagogic competence will be strengthened immensely. Such systematic sharing of experience and expertise can be of great help.

6. Curricular choices

Acknowledging the existence of choices in curriculum is an important step in the institutionalisation of education. Hence, when we speak of shifting the focus from content to learning environments, we are offering criteria by which a curriculum designer may resolve choices. For instance, visualisation and geometric reasoning are important processes to be ensured, and this has implications for teaching algebra. Students who 'blindly' manipulate equations without being able to visualise and understand the underlying geometric picture cannot be said to have understood. If this means greater coverage for geometric reasoning (in terms of lessons, pages in textbook), it has to be ensured. Again, if such expansion can only be achieved by reducing other (largely computational) content, such content reduction is implied.

Below, while discussing stage-wise content, we offer many such inclusion /exclusion criteria for the curriculum designer, emphasising again that the recommendation is not to dilute content, but to give importance to a variety of processes. Moreover, we suggest a principle of postponement: in general, if a theme can be offered with better motivation and applications at a later stage, wait for introducing it at that stage, rather than go for technical preparation without due motivation. Such considerations are critical at the secondary and higher secondary stages where a conscious choice between breadth and depth is called for. Here, a quotation from William Thurston is appropriate:

The long-range objectives of mathematics education would be better served if the tall shape of mathematics were de-emphasised, by moving away from a standard sequence to a more diversified curriculum with more topics that start closer to the ground. There have been some trends in this direction, such as courses in finite mathematics and in probability, but there is room for much more.17

6.1 Primary Stage

Any curriculum for primary mathematics must incorporate the progression from the concrete to the abstract, and subsequently, a need to appreciate the importance of abstraction in mathematics. In the lowest classes, especially, it is important that activities with concrete objects form the first step in the classroom to enable the child to understand the connections between the logical functioning of their everyday lives to that of mathematical thinking.

Mathematical games, puzzles and stories involving numbers are useful to enable children to make these connections and to build upon their everyday understandings. Games – not to be confused with open-ended play - provide non-didactic feedback to the child, with a minimum amount of teacher intervention[18]. They promote processes of anticipation, planning and strategy.

6.1.1 Mathematics is not just arithmetic

While addressing number and number operations, due place must be given to non-number areas of mathematics. These include shapes, spatial understanding, patterns, measurement and data handling. It is not enough to deal with shapes and their properties as a prelude to geometry in the higher classes. It is important also to build up a vocabulary of relational words which extend the child's understanding of space. The identification of patterns is central to mathematics. Starting with simple patterns of repeating shapes, the child can move on to more complex patterns involving shapes as well as numbers. This lays the base for a mode of thinking that can be called algebraic. A primary curriculum that is rich in such activities can arguably make the transition to algebra easier in the middle grades[19]. Data handling, which forms the base for statistics in the higher classes, is another neglected area of school mathematics and can be introduced right from Class I.

6.1.2 Number and number operations

Children come equipped with a set of intuitive and cultural ideas about number and simple operations at the point of entry into school. These should be used to make linkages and connections to number understanding rather than treating the child as a *tabula rasa*. To learn to think in mathematical ways, children need to be logical and to understand logical rules, but they also need to learn conventions needed for the mastery of mathematical techniques such as the use of a base ten system. Activities as basic as counting and understanding numeration systems involve logical understandings for which children need time and practice if they are to attain mastery, and then to be able to use them as tools for thinking and for mathematical problem solving[20].

Working with limited quantities and smaller numbers prevents overloading the child's cognitive capacity which can be better used for mastering the logical skills at these early stages.

Operations on natural numbers usually form a major part of primary mathematics syllabi. However, the standard algorithms of addition, subtraction, multiplication and division of whole numbers in the curriculum have tended to occupy a dominant role in these. This tends to happen at the expense of development of number sense and skills of estimation and approximation. The result frequently is that students, when faced with word problems, ask "Should I add or subtract?, Should I multiply or divide?" This lack of a conceptual base continues to haunt the child in later classes. All this strongly suggests that operations should be introduced contextually. This should be followed by the development of language and symbolic notation, with the standard algorithms coming at the end rather than the beginning of the treatment.

6.1.3 Fractions and decimals

Fractions and decimals constitute another major problem area. There is some evidence that the introduction of operations on fractions coincides with the beginnings of fear of mathematics. The content in these areas needs careful reconsideration. Everyday contexts in which fractions appear, and in which arithmetical operations need to be done on them, have largely disappeared with the introduction of metric units and decimal currency. At present, the child is presented with a number of contrived situations in which operations have to be performed on fractions. Moreover, these operations have to be done using a set of rules which appear arbitrary (often, even to the teacher), and have to be memorised - this at a time when the child is still grappling with the rules for operating on whole numbers. While the importance of fractions in the conceptual structure of mathematics is undeniable, the above considerations seem to suggest that less emphasis on operations with fractions at the primary level is called for.

6.2 Upper Primary Stage

Mathematics is amazingly compressible: one may struggle a lot, work out something, perhaps by trying many methods. But once it is understood, and seen as a whole, it can be filed away, and used as just a step when needed. The insight that goes into this compression is one of the great joys of mathematics. A major goal of the upper primary stage is to introduce the student to this particular pleasure.

The compressed form lends itself to application and use in a variety of contexts. Thus, mathematics at this stage can address many problems from everyday life, and offer tools for addressing them. Indeed, the transition from arithmetic to algebra, at once both challenging and rewarding, is best seen in this light.

6.2.1 Arithmetic and Algebra

A consolidation of basic concepts and skills learnt at primary school is necessary from several points of view. For one thing, ensuring numeracy in all children is an important aspect of universalisation of elementary education. Secondly, moving from number sense to number patterns, seeing relationships between numbers and looking for patterns in the relationships bring useful life skills to children. Ideas of prime numbers, odd and even numbers, tests of divisibility etc. offer scope for such exploration.

Algebraic notation, introduced at this stage, is best seen as a compact language, a means of succinct expression. Use of variables, setting up and solving linear equations, identities and factoring are means by which students gain fluency in using the new language.

The use of arithmetic and algebra in solving real problems of importance to daily life can be emphasised. However, engaging children's interest and offering a sense of success in solving such problems is essential.

6.2.2 Shape, Space and Measures

A variety of regular shapes are introduced to students at this stage: triangles, circles, quadrilaterals, they offer a rich new mathematical experience in at least four ways. Children start looking for such shapes in nature, all around them, and thereby discover much symmetry and acquire a sense of aesthetics. Secondly, they learn how many seemingly irregular shapes can be approximated by regular ones, which becomes an important technique in science. Thirdly, they start comprehending the idea of space: for instance, that a circle is a path or boundary which separates the space inside the circle from that outside it. Fourthly, they start associating numbers with shapes, like area, perimeter etc, and this technique of quantisation, or arithmetisation, is of great importance. This also suggests that mensuration is best when integrated with geometry.

An informal introduction to geometry is possible using a range of activities like paper folding and dissection, and exploring ideas of symmetry and transformation. Observing geometrical properties and inferring geometrical truth is the main objective here. Formal proofs can wait for a later stage.

6.2.3 Visual Learning

Data handling, representation and visualisation are important mathematical skills which can be taught at this stage. They can be of immense use as 'life skills'. Students can learn to appreciate how railway time tables, directories and calendars organise information compactly.

Data handling should be suitably introduced as tools to understand process, represent and interpret day-to-day data. Use of graphical representations of data can be encouraged. Formal techniques for drawing linear graphs can be taught.

Visual Learning fosters understanding, organisation and imagination. Instead of emphasising only two-column proofs, students should also be given opportunities to justify their own conclusions with less formal, but nonetheless convincing, arguments. Students' spatial reasoning and visualisation skills should be enhanced. The study of geometry should make full use of all available technology. A student when given visual scope to learning, remembers pictures, diagrams, flowcharts, formulas and procedures.

6.3 Secondary Stage

It is at this stage that Mathematics comes to the student as an academic discipline. In a sense, at the elementary stage, mathematics education is (or ought to be) guided more by the logic of children's psychology of learning rather than the logic of mathematics. But at the secondary stage, the student begins to perceive the structure of mathematics. For this, the notions of *argumentation* and proof become central to curriculum now.

Mathematical terminology is highly stylised, self-conscious and rigorous. The student begins to feel comfortable and at ease with the characteristics of mathematical communication: carefully defined terms and concepts, the use of symbols to represent them, precisely stated propositions using only terms defined earlier, and proofs justifying propositions. The student appreciates how an edifice is built up, arguments constructed using propositions justified earlier, to prove a theorem, which in turn is used in proving more.

For long, geometry and trigonometry have wisely been regarded as the arena wherein students can learn to appreciate this structure best. In the elementary stage, if students have learnt many shapes and know how to associate quantities and formulas with them, here they start reasoning about these shapes using the defined quantities and formulas.

Algebra, introduced earlier, is developed at some length at this stage. Facility with algebraic manipulation is essential, not only for applications of mathematics, but also internally in mathematics. Proofs in geometry and trigonometry show the usefulness of algebraic machinery. It is important to ensure that students learn to geometrically visualise what they accomplish algebraically.

A substantial part of the secondary mathematics curriculum can be devoted to consolidation. This can be and needs to be done in many ways. Firstly, the student needs to integrate the many techniques of mathematics she has learnt into a problem solving ability. For instance, this implies a need for posing problems to students which involve more than one content area: algebra and trigonometry, geometry and mensuration, and so on. Secondly, mathematics is used in the physical and social sciences, and making the connections explicit can inspire students immensely. Thirdly, mathematical modelling, data analysis and interpretation, taught at this stage, can consolidate a high level of literacy. For instance, consider an environment-related project, where the student has to set up a simple linear approximation and model a phenomenon, solve it, visualise the solution, and deduce a property of the modelled system. The consolidated learning from such an activity builds a responsible citizen, who can later intuitively analyse information available in the media and contribute to democratic decision making.

At the secondary stage, a special emphasis on experimentation and exploration may be worthwhile. *Mathematics laboratories* are a recent phenomenon, which hopefully will expand considerably in future[22]. Activities in practical mathematics help students immensely in visualisation.

Indeed, Singh, Avtar and Singh offer excellent suggestions for activities at all stages. Periodic systematic evaluation of the impact of such laboratories and activities[23] will help in planning strategies for scaling up these attempts.

6.4 Higher Secondary Stage

Principally, the higher secondary stage is the launching pad from which the student is guided towards career choices, whether they imply university education or otherwise. By this time, the student's interests and aptitude have been largely determined, and mathematics education in these two years can help in sharpening her abilities.

The most difficult curricular choice to be made at this stage relates to that between breadth and depth. A case can be made for a broad based curriculum that offers exposure to a variety of subjects; equally well, we can argue for limiting the number of topics to a few and developing competence in the selected areas. While there are no formulaic answers to this question, we point to the Thurston remark quoted above once again.

Indeed, Thurston is in favour of breadth even as an alternative to remedial material which merely goes over the same material once more, handicapping enthusiasm and spontaneity.

Instead, there should be more courses available … which exploit some of the breadth of mathematics, to permit starting near the ground level, without a lot of repetition of topics that students have already heard.

When we choose breadth, we not only need to decide which themes to develop, but also how far we want to go in developing those themes. In this regard, we suggest that the decision be dictated by *mathematical considerations*. For instance, introd-ucing projective geometry can be more important for mathematics as a discipline than projectile motion (which can be well studied in physics). Similarly, the length of treatment should be dictated by whether *mathematical objectives* are met. For instance, if the objective of introducing complex numbers is to show that the enriched system allows for solutions to all polynomial equations, the theme should be developed until the student can at least get an idea of how this is possible. If there is no space for such a treatment, it is best that the theme not be introduced; showing operations on complex numbers and representations without any understanding of why such a study is relevant is unhelpful.

Currently, mathematics curriculum at the higher secondary stage tends to be dominated by differential and integral calculus, making for more than half the content in Class XII. Since Board examinations are conducted on Class XII syllabus, this subject acquires tremendous importance among students and teachers. Given the nature of Board examinations as well as other entrance examinations, the manipulative and computational aspects of calculus tend to dominate mathematics at this stage. This is a great pity, since many interesting topics (sets, relations, logic, sequences and series, linear inequalities, combinatorics) introduced to students in Class XI can give them good mathematical insight but these are typically given short shrift. Curriculum designers should address this problem while considering the distribution of content between Classes XI and XII.

In many parts of the world, the desirability of having electives at this stage, offering different aspects of mathematics, has been acknowledged. However, implementation of a system of electives is dauntingly difficult, given the need for a variety of textbooks and more teachers, as well as the centralised nature of examinations. Yet, experimenting with ideas that offer a range of options to students will be worthwhile.

6.5 Mathematics and Mathematicians

At all stages of the curriculum, an element of humanising the curriculum is essential. The development of mathematics has many interesting stories to be told, and every student's daily life includes many experiences relevant to mathematics. Bringing these stories and accounts into the curriculum is essential for children to see mathematics in perspective. Lives of mathematicians and stories of mathematical insights are not only endearing, they can also be inspiring.

A specific case can be made for highlighting the contribution made by Indian mathematicians. An appreciation of such contributions will help students see the place of mathematics in our culture. Mathematics has been an important part of Indian history and culture, and students can be greatly inspired by understanding the seminal contributions made by Indian mathematicians in early periods of history.

Similarly, contributions by women mathematicians from all over the world are worth highlighting. This is important, mainly to break the prevalent myth that mathematics has been an essentially male domain, and also to invite more girls to the mathematical enterprise.

7. Conclusion

In a sense, all these are steps advocated by every mathematics educator over decades. The difference here is in emphasis, in achieving these actions by way of curricular choices. Perhaps the most compelling reason for the vision of mathematics education we have articulated is that our children will be better served by higher expectations, by curricula which go far beyond basic skills and include a variety of mathematical models, and by pedagogy which devotes a greater percentage of instructional time to problem solving and active learning. Many students respond to the current curriculum with boredom and discouragement, develop the perception that success in mathematics depends on some innate ability which they simply do not have, and feel that, in any case, mathematics will never be useful in their lives. Learning environments like the one described in the vision will help students to enjoy and appreciate the value of mathematics, to develop the tools they need for varied educational and career options, and to function effectively as citizens.

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