DANGER OF IMPROPER USE OF MATHEMATICAL RESULTS, FORMULAS AND SYMBOLS : TRIGONOMETRY

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The mistakes appearing in mathematics textbooks may often lead to defective teaching. Some of these common mistakes are conceptual mistakes in solving problems, in the statement of mathematical formulas or in steps involved in derivations. Often, the teachers fall victim to these errors and incorrectly solve a problem in the class, say by improper use of a formula, theorem, etc. This, in turn, perpetuates wrong ideas among young children. The author has selected a few of such mistakes and errors in trigonometry at the higher secondary stage. These examples are based on first-hand experience of the author with textbooks and classroom teaching practices for many years. In this article he has cited many interesting examples to explain some of these errors and mistakes in the teaching of trigonometry.

Trigonometry plays a very important part in mathematics. Students and teachers solve problems of trigonometry perhaps without much difficulty by using a number of trigonometric formulas and results. When they get the correct answer, they are fully satisfied, but when they get some incorrect answer or two or more answers for the same problem by using different techniques, they fail to give proper reasonings. They hardly think about the validity of these formulas and results in solving problems as a result of which they sometimes arrive at a conclusion which is physically impossible. In the actual classroom situation, students and teachers may sometimes come across this type of fallacious results for which they may find it difficult to get proper explanation. It should be noted that any fallacious result in any branch of mathematics may be due to the violation of some rules of mathematics. Some theorems, formulas or results in mathematics may be true under certain restrictions or conditions. In practical situation, we memorise these formulas, theorems, etc., giving the least importance of these restrictions and conditions.

Below an attempt is made to show with some examples that there is a danger of violating these restrictions and conditions in trigonometric formulas and results.

Equalisation of Arguments

Students have the tendency to equalise the arguments from a trigonometric equation. Thus if tan x = tan y, they conclude that x = y. That it is not always true will be clear from the following example:

$\tan x = \tan(\pi + x)$

$\therefore x = \pi + x \therefore \pi = 0$, a result which is absurd. The explanation is that all trigonometric functions are periodic functions. In the case of periodic functions, equalisation of arguments is not always permissible. In fact, equalisation of arguments from a trigonometric equation is allowed only when both the angles lie in the interval $(0, \pi/2)$.

Thus if $0 \leq A, B \leq \pi/2$,

then $\sin A = \sin B \Rightarrow A = B$,

$\tan A = \tan B \Rightarrow A = B$,

and so on.

Choosing the Scale in Drawing Graphs of Trigonometric Functions

In elementary mathematics, arguments in all trigonometric ratios are expressed either in radians or in degrees. In $y = \sin x$, for example, the unit of x is in radian. But $\sin x$, i.e., y is a number.

In drawing the graph of say, $y = \sin x$, we usually choose the same scale on x-axis as in y-axis. The question is: Is it necessary to choose the same scale on both x and y axes? The answer is 'no' if x in $\sin x$ has a unit. This is because we represent numbers on y-axis and radians on x-axis. If one unit on a graph paper along y-axis represents 1, there is no reason to believe that the same 1 unit along x-axis will represent 1 radian. We are free to choose any number of units of the graph paper along the x-axis as 1 radian. In physics and chemistry, we come across with a lot of equations connecting two variables with different units, and in drawing the graphs we choose different scales along x and y-axes. For example, in Boyle's law equation: PV = constant, P has the unit of pressure and V has the unit of volume and hence in drawing the graph, we are free to choose different scales for P and V.

Restrictions Involved in Inverse Trigonometric Functions

It is known that every inverse trigonometric function is single-valued and hence is defined within its principal value branch. We use well-known formulas of inverse trigonometric functions without paying much attention to the possible restrictions for which these are valid. As a result, we may arrive at some fallacious results for which we may not find any explanation. These are discussed below with series of examples.

Example 1

Students are familiar with the well-known formula:

$$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x + y}{1 - xy}.$$

If we now put x = 1 and y = 2, then L.H.S. = $\tan^{-1}1 + \tan^{-1}2$ which is greater than 0. Also R.H.S. = $\tan^{-1}\left(\frac{3}{-1}\right) = \tan^{-1}(-3) = -\tan^{-1}3$ which is less than 0.

It follows that a negative non-zero number is equal to a positive non-zero number which is absurd. Why is it so if the above formula is correct? This is because the above formula is true under certain values of x and y such that $xy < 1$. If we neglect this restriction, we are likely to get the absurd result as shown above.

Example 2

The formula:

$2\tan^{-1}x = \sin^{-1}\frac{2x}{1 + x^2}$ is also well-known.

When we use this formula in solving some problems, we do not generally bother about whether it is true for all x or not. As a result, we again come across some absurd results. It can be shown that the above formula is true only for $-1 \leq x \leq 1$. For, if we take, say $x = \sqrt{3}$, (>1), then

L.H.S. = $2\tan^{-1}$“3 = $2\pi/3$, whereas

$$\text{R.H.S.} = \sin^{-1}\left(\frac{2\sqrt{3}}{1+3}\right) = \sin^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{3}$$

Hence, L.H.S. $\neq$ R.H.S. for $x = \sqrt{3}$.

Example 3

We now consider another well-known formula, viz.,

$$2\tan^{-1}x = \cos^{-1}\frac{1-x^2}{1+x^2}$$

This formula is also used without paying heed to the possible restrictions on x. That the result is not true for $x < 0$, can be shown by taking, say, $x = -\sqrt{3}$.

Then L.H.S. = $2\tan^{-1}(-\sqrt{3}) = -2\pi/3$

$$\text{R.H.S.} = \cos^{-1}\frac{1-3}{1+3} = \cos^{-1}\left(-\frac{1}{2}\right)$$

$= \pi - \cos^{-1}1/2 = \pi - \pi/3 = 2\pi/3$

Hence, L.H.S. $\neq$ R.H.S. for $x = -\sqrt{3}$

In fact, the given result is true for $0 \leq x < \infty$

Example 4

The result $2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}$ is another important result which is true for $-1 < x < 1$. For $x = \pm 1$, the L.H.S. is defined whereas the R.H.S. is undefined. That the result is not true beyond the interval $-1 < x < 1$ can be shown by putting $x = \sqrt{3}$ on both sides.

Example 5

We very often use the following formula without imposing any restriction on x :

$\cot^{-1}x = \tan^{-1}1/x$, $\sec^{-1}x = \cos^{-1}1/x$, $\text{cosec}^{-1}x = \sin^{-1}1/x$.

In fact, the above results are true for $x \neq 0$. For $x = 0$, R.H.S. for each of the above equations is undefined whereas the L.H.S. is defined for $x = 0$.

Example 6

If we do not consider these restrictions on x while memorising the important formulas as mentioned above, we are likely to get some unexpected results. As an example,

$\sin^{-1}\left(2x\sqrt{1-x^2}\right)$ can be shown to be equal to $2\sin^{-1}x$ as well as $2\cos^{-1}x$ which are not identical functions.

We first put $x = \sin\theta$.

Then $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = \sin^{-1}(2\sin\theta\cos\theta)$

$= \sin^{-1}\sin 2\theta = 2\theta = 2\sin^{-1}x$, ...(1)

Again, we put $x = \cos\theta$.

Then $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = \sin^{-1}(2\sin\theta\cos\theta) = \sin^{-1}\sin 2\theta = 2\theta = 2\cos^{-1}x$. ....(2)

Why are we getting two different answers?

The explanation is that the result (1) is true when

$-1/\sqrt{2} \le x \le 1/\sqrt{2}$, whereas the result (2) is true when $0 \le x \le 1$.

Hence, both the results are true, but for different values of x. To explain how these two different sets of values of x are obtained in the two cases, we recall that the principal value branches of $\sin^{-1} x$ and $\cos^{-1} x$ are respectively $-\pi/2 \le \sin^{-1} x \le \pi/2$ and $0 \le \cos^{-1} x \le \pi/2$.

Now from (1), x should be such that

$-\pi/2 \le \sin^{-1}\left(2x\sqrt{1-x^2}\right) \le \pi/2$

or, $-\pi/2 \le 2\sin^{-1} x \le \pi/2$

or, $-\pi/4 \le \sin^{-1} x \le \pi/4$

or, $-1/\sqrt{2} \le x \le 1/\sqrt{2}$.

In the case of the equation (2), we note that the principal value branch of $\cos^{-1} x \ge 0$, but the principal value branch of $\sin^{-1}\left(2x\sqrt{1-x^2}\right)$ may be negative also. Hence, taking the common region for which both the functions are defined, we have

$0 \le 2\cos^{-1} x \le \pi$

or, $0 \le \cos^{-1} x \le \pi/2$

or, $0 \le x \le 1$.

Example 7

Another example of getting a wrong answer to a problem due to the violation of the possible restriction on x while using a formula as mentioned above can be given.

Suppose, we are asked to solve the following equation:

$$\tan^{-1}\frac{2x}{1-x^2} + \cot^{-1}\frac{1-x^2}{2x} = \frac{\pi}{3}$$

The solution of the above equation can be obtained in the following manner:

From the given equation, we have

$$\tan^{-1}\frac{2x}{1-x^2} + \tan^{-1}\frac{2x}{1-x^2} = \frac{\pi}{3}$$

or, $2\tan^{-1}\dfrac{2x}{1-x^2} = \dfrac{\pi}{3}$

or, $\tan^{-1}\dfrac{2x}{1-x^2} = \dfrac{\pi}{6}$ (A)

If we now use the formula as stated in example 4 above, we get

$2\tan^{-1}x = \dfrac{\pi}{6}$

or, $\tan^{-1}x = \dfrac{\pi}{12}$

or, $x = \tan\dfrac{\pi}{12} = 2 - \sqrt{3}$

If we, however, do not use the above formula, then from (A), we get

$\dfrac{2x}{1-x^2} = \tan\dfrac{\pi}{6} = 1/\sqrt{3}$

or, $x^2 + 2\sqrt{3}x - 1 = 0$

or, $x = \dfrac{-2\sqrt{3} \pm \sqrt{12+4}}{2} = -\sqrt{3} \pm 2$

i.e., the values of x are $-\sqrt{3} + 2$ and $-\sqrt{3} - 2$. We, thus, see that due to the application of a formula, we lose some solution. Both the solutions obtained by avoiding the formula satisfy the original equation and hence these are the actual complete solution of the given equation.

The incomplete solution obtained as a result of the application of formula

$$2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}$$

is due to the fact that the above formula is not true for all values of x. It is true only when $-1 < x < 1$ and that is why we did not get the solution $-\sqrt{3}-2$ which is obviously less than -1. This example shows that if we use a formula at random without thinking about the possible restrictions, we are likely to obtain incomplete and sometimes wrong answers.

Sometimes, due to the wrong use of a formula or due to the violation of possible values of x for which the formula holds, we get contradictory results. We consider an example:

Example 8

Simplify the expression:

$$\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)$$

We solve the problem in several ways:

First Method

Writing $\sqrt{1+\sin x} = \sin\frac{x}{2}+\cos\frac{x}{2}$

and $\sqrt{1-\sin x} = \sin\frac{x}{2}-\cos\frac{x}{2}$, we get the given expression:

$$= \cot^{-1}\left(\frac{\sin\frac{x}{2}+\cos\frac{x}{2}+\sin\frac{x}{2}-\cos\frac{x}{2}}{\sin\frac{x}{2}+\cos\frac{x}{2}-\sin\frac{x}{2}+\cos\frac{x}{2}}\right)$$

$$= \cot^{-1}\tan\frac{x}{2} = \cot^{-1}\cot\left(\frac{\pi}{2}-\frac{x}{2}\right) = \frac{\pi}{2}-\frac{x}{2} \qquad \text{..... (B)}$$

Second Method

If we write

$\sqrt{1+\sin x} = \cos\frac{x}{2}+\sin\frac{x}{2}$

and $\sqrt{1-\sin x} = \cos\frac{x}{2}-\sin\frac{x}{2}$, then the given expression

$$= \cot^{-1}\left(\frac{\cos\frac{x}{2}+\sin\frac{x}{2}+\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}-\cos\frac{x}{2}+\sin\frac{x}{2}}\right)$$

$$= \cot^{-1}\left(\frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}\right) = \cot^{-1}\cot\frac{x}{2} = \frac{x}{2} \qquad \text{............... (C)}$$

Third Method

If we rationalise the denominator, then the given expression

$$= \cot^{-1}\left(\frac{\left(\sqrt{1+\sin x}+\sqrt{1-\sin x}\right)^2}{1+\sin x-1+\sin x}\right)$$

$$= \cot^{-1}\left(\frac{1+\sin x+1-\sin x+2\sqrt{1-\sin^2 x}}{2\sin x}\right)$$

$$= \cot^{-1}\left(\frac{1+\cos x}{\sin x}\right)$$

$$= \cot^{-1}\left(\frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\right)$$

$= \cot^{-1}\cot\frac{x}{2}$

$= \frac{x}{2}$ .... (D)

We observe that the results (C) and (D) are the same but the result (B) is different. Why is it so? It is because in all the three methods of solution, the trigonometric formulas have been randomly used without giving a thought to the possible restrictions on x.

In the first two methods, for example, the results:

$\sqrt{1 \pm \sin x} = \sin\frac{x}{2} \pm \cos\frac{x}{2}$

and $\sqrt{1 \pm \sin x} = \cos\frac{x}{2} \pm \sin\frac{x}{2}$

have been used without considering the signs of $\sin\frac{x}{2} \pm \cos\frac{x}{2}$ and $\cos\frac{x}{2} \pm \sin\frac{x}{2}$.

We know that:

$\sqrt{x^2} = x$, if $x \geq 0$

$= -x$, if $x \leq 0$.

Hence, $\sqrt{1 + \sin x} = \sqrt{\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)^2}$

$= \sin\frac{x}{2} + \cos\frac{x}{2}$, if $\sin\frac{x}{2} + \cos\frac{x}{2} \geq 0$

$= -\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)$, if $\sin\frac{x}{2} + \cos\frac{x}{2} \leq 0$

Now, $\sin\frac{x}{2} + \cos\frac{x}{2} \geq 0$

$\Rightarrow \cos\left(\frac{x}{2} - \frac{\pi}{4}\right) \geq 0$

$\Rightarrow -\frac{\pi}{2} \leq \frac{x}{2} - \frac{\pi}{4} \leq \frac{\pi}{2}$

$\Rightarrow -\frac{\pi}{2} \leq x \leq \frac{3\pi}{2}$

Similarly, $\sin\frac{x}{2} + \cos\frac{x}{2} \leq 0$

$\Rightarrow \cos\left(\frac{x}{2} - \frac{\pi}{4}\right) \leq 0$

$\Rightarrow \frac{\pi}{2} \leq \frac{x}{2} - \frac{\pi}{4} \leq \frac{3\pi}{2}$

$\Rightarrow \frac{3\pi}{2} \leq x \leq \frac{7\pi}{2}$

Again, $\sqrt{1 - \sin x} = \sqrt{\left(\sin\frac{x}{2} - \cos\frac{x}{2}\right)^2}$

$= \sin\frac{x}{2} - \cos\frac{x}{2}$, if $\sin\frac{x}{2} - \cos\frac{x}{2} \geq 0$

$= \cos\frac{x}{2} - \sin\frac{x}{2}$, if $\cos\frac{x}{2} - \sin\frac{x}{2} \geq 0$

Now, $\sin\frac{x}{2} - \cos\frac{x}{2} \geq 0$

$\Rightarrow \cos\left(\frac{x}{2} + \frac{\pi}{4}\right) \leq 0$

$\Rightarrow \frac{\pi}{2} \leq \frac{x}{2} + \frac{\pi}{4} \leq \frac{3\pi}{2}$

$\Rightarrow \frac{\pi}{4} \leq \frac{x}{2} \leq \frac{5\pi}{4}$

$\Rightarrow \frac{\pi}{2} \leq x \leq \frac{5\pi}{2}$

Similarly,

$\cos\frac{x}{2} - \sin\frac{x}{2} \geq 0$

$\Rightarrow \cos\left(\frac{x}{2} + \frac{\pi}{4}\right) \geq 0$

$$\Rightarrow -\frac{\pi}{2} \le \frac{x}{2} + \frac{\pi}{4} \le \frac{\pi}{2}$$

$$\Rightarrow -\frac{3\pi}{4} \le \frac{x}{2} \le \frac{\pi}{4}$$

$$\Rightarrow -\frac{3\pi}{2} \le x \le \frac{\pi}{2}$$

We thus have the following results:

$$\sqrt{1+\sin x} = \sin\frac{x}{2} + \cos\frac{x}{2}, \quad \text{if } -\frac{\pi}{2} \le x \le \frac{3\pi}{2} \quad \text{....(i)}$$

$$= -\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right), \quad \text{if } \frac{3\pi}{2} \le x \le \frac{7\pi}{2} \quad \text{....(ii)}$$

$$\sqrt{1-\sin x} = \sin\frac{x}{2} - \cos\frac{x}{2}, \quad \text{if } \frac{\pi}{2} \le x \le \frac{5\pi}{2} \quad \text{....(iii)}$$

$$= \cos\frac{x}{2} - \sin\frac{x}{2}, \quad \text{if } -\frac{3\pi}{2} \le x \le \frac{\pi}{2} \quad \text{....(iv)}$$

From (i) and (ii),

$$\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}} = \tan\frac{x}{2}, \quad \text{if } -\frac{\pi}{2} \le x \le \frac{3\pi}{2}$$

and $\frac{\pi}{2} \le x \le \frac{5\pi}{2}$

$$\Rightarrow \frac{\pi}{2} \le x \le \frac{3\pi}{2}$$

In terms of general values,

$$2n\pi + \frac{\pi}{2} \le x \le 2n\pi + \frac{3\pi}{2} \quad (n = 0, \pm 1, \pm 2, \ldots\ldots\ldots\ldots\ldots)$$

From (ii) and (iv),

$$\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}} = \tan\frac{x}{2}, \quad \text{if } \frac{3\pi}{2} \le x \le \frac{7\pi}{2}$$

and $-\frac{3\pi}{2} \le x \le \frac{\pi}{2}$

which is absurd.

From (i) and (iv),

$$\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}} = \cot\frac{x}{2}, \quad \text{if } -\frac{\pi}{2} \le x \le \frac{3\pi}{2}$$

and $-\frac{3\pi}{2} \le x \le \frac{\pi}{2}$

$$\Rightarrow -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

In terms of general values,

$$2n\pi - \frac{\pi}{2} \le x \le 2n\pi + \frac{\pi}{2} \quad (n = 0, \pm 1, \pm 2, \ldots\ldots)$$

From (ii) and (iii),

$$\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}} = \cot\frac{x}{2}, \quad \text{if } \frac{3\pi}{2} \le x \le \frac{7\pi}{2}$$

and $\frac{\pi}{2} \le x \le \frac{5\pi}{2}$

$$\Rightarrow \frac{3\pi}{2} \le x \le \frac{5\pi}{2}$$

which is included in the general values of x given above.

We thus have

$$\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)$$

$$= \cot^{-1}\tan\frac{x}{2}$$

$$= \cot^{-1}\cot\left(\frac{\pi}{2}-\frac{x}{2}\right)$$

$$= \frac{\pi}{2} - \frac{x}{2}, \quad \text{if } 2n\pi + \frac{\pi}{2} \le x \le 2n\pi + \frac{3\pi}{2}$$

and the above expression

$$= \cot^{-1}\cot\frac{x}{2} = \frac{x}{2}, \quad \text{if } 2n\pi - \frac{\pi}{2} \le x \le 2n\pi + \frac{\pi}{2}.$$

We thus see that although both the results shown in the first and the second methods are true, they are valid for different sets of values of x. We can get the same sets of values of x, if we solve the original problem in the third method in the following manner:

$$\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)$$

$$=\cot^{-1}\left(\frac{1+\sqrt{\cos^2 x}}{\sin x}\right)$$ (Rationalising the denominator as in the third method)

$$=\cot^{-1}\left(\frac{1+\cos x}{\sin x}\right), \quad \text{if}$$

$$\cos x \geq 0 \Rightarrow 2n\pi-\frac{\pi}{2} \leq x \leq 2n\pi+\frac{\pi}{2}$$

$$=\cot^{-1}\cot\frac{x}{2}=\frac{x}{2}, \quad \text{if } 2n\pi-\frac{\pi}{2} \leq x \leq 2n\pi+\frac{\pi}{2},$$

If, however,

$$\cos x \leq 0 \Rightarrow 2n\pi+\frac{\pi}{2} \leq x \leq 2n\pi+\frac{3\pi}{2},$$

$$\text{then } \cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)$$

$$=\cot^{-1}\left(\frac{1-\cos x}{\sin x}\right)=\cot^{-1}\left(\frac{2\sin^2\frac{x}{2}}{2\sin\frac{x}{2}.\cos\frac{x}{2}}\right)$$

$$=\cot^{-1}\tan\frac{x}{2}=\cot^{-1}\cot\left(\frac{\pi}{2}-\frac{x}{2}\right)=\frac{\pi}{2}-\frac{x}{2}$$

Thus we have obtained the same results as before. It is now clear that if we are not careful enough about the validity of some trigonometrical formulas in solving a problem, we are likely to get different answers if we solve the problem by different methods and we shall be unable to explain the cause of these different answers. Of course, there are many problems in which we get a unique solution in whatever methods we solve the problems even if we do not bother about the validity of the formulas etc. For example, to simplify an expression like

$$\sin^{-1}\left(x\sqrt{1-x}-\sqrt{x}\sqrt{1-x^2}\right),$$

we can solve the problem in various ways. We can substitute

(i) $x=\sin\theta$ and $\sqrt{x}=\cos\phi$ or

(ii) $x=\cos\theta$ and $\sqrt{x}=\sin\phi$ or

(iii) $x=\sin\theta$ and $\sqrt{x}=\sin\phi$ or

(iv) $x=\cos\theta$ and $\sqrt{x}=\cos\phi$

In each case we shall get the final answer as $\sin^{-1}x-\sin^{-1}\sqrt{x}$. But it is always safe to think about the possible values of x. For example, in the above problem, in order that the given expression is real, it is necessary that $0 \leq x \leq 1$. Further, θ, ϕ, as introduced in (i) to (iv) must be such that $0 \leq \theta, \phi \leq \pi/2$.

In conclusion, it may be stated that the examples, which have been given above are only a few out of many fallacious results which may arise due to inappropriate use of the formulas, results, etc., in trigonometry and it is hoped that these will be of some benefit to the readers.

Reference

Das, S.C. 1980. 'Danger of improper use of mathematical results, formulas and symbols: Algebra'. *School Science*. XVIII (1), March, pp. 40.