LEARNING THROUGH RIDDLES

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The Throne Carpet

King Vikram Singh was a great lover of art. He encouraged artists, not only those who lived in his kingdom but also those who came from abroad. Once it so happened that a great Kashmiri artist, named Rafi, appeared in his court and showed the king a variety of beautiful carpets he had woven. The king was greatly pleased and ordered Rafi to make a beautiful carpet to cover his throne. Rafi gladly agreed to do so and carefully studied the throne. The first thing which he noted was that the length of each step of the throne was equal but their widths and heights were different, as shown in Fig.1.

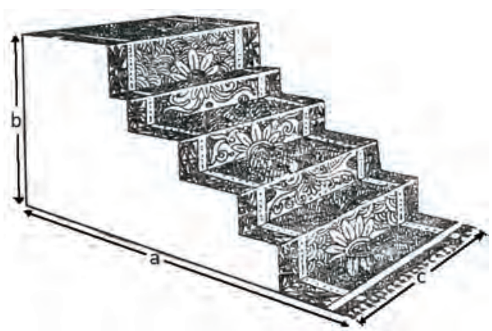

Fig. 1

Rafi, naturally, wanted the various measurements of the throne to know the size of the carpet to be made. Now, according to the royal tradition of the kingdom no one except the king could put his foot on the throne. Also, Rafi could not ask the king himself to measure the length, width and height of each of the steps. He also knew that failure in designing a properly fitting carpet would mean death. A worried Rafi took the king's permission to leave the court; went to the room where he was staying. For the whole night he pondered over the problem and thought of a solution when the cock crowed. He then happily retired to his bed. The following day he came to court and simply measured the lengths a, b and c (see Fig.2). He found a to be 18 ft, b to be 15 ft and c to 6 ft. He then designed a carpet 18+15 = 33 ft long and kept the width of the carpet 6 ft. When he presented the carpet, thus designed, to the king, all the courtiers including the king himself were astonished to see how exactly the beautiful carpet fitted the throne.

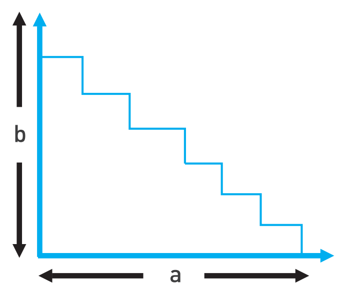

Fig. 2

How did it happen? Which principle helped Rafi in concluding that the length of the needed carpet is simply given by a+b? Let us see. For this purpose look at Fig. 3, where 1 (A S)= b and 1(A G)= a.

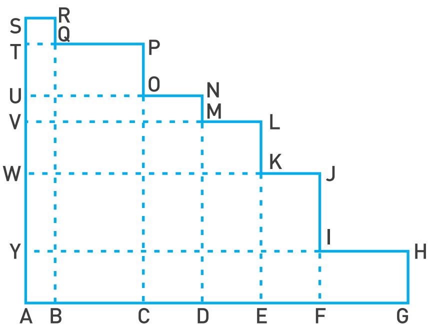

Fig.3

We know that opposite sides of any rectangle are congruent. Hence from Fig. 3 we can say that

$SR \cong AB$

$QP \cong BC$

$ON \cong CD$

$ML \cong DE$

$KJ \cong EF$

$IH \cong FG$

Hence the sum of the lengths of line segments on the left side will be equal to the sum of lengths of the line segments on the right side.

Hence

1 (S R) + 1 (Q P) + 1 (O N) + 1 (M L) + 1 (K L) + 1 (I H)

= 1 (A B) + 1 (B C) + 1 (C D) + 1 (D E) + 1 (E F) + 1 (F G)

= 1(A G) = a..................1.

Similarly,

1 (R Q) = 1 (S T)

1 (P O) = 1 (T U)

1 (N M) = 1 (U V)

1 (L K) = 1 (V W)

1 (J I) = 1 (W Y)

1 (H G) = 1 (Y A)

Hence, adding, we get

1 (R Q) + 1(P O) + 1(N M) +1(L K) +1(J I) +1(H G)=1(S T) +1(T U) +1(U V)+ 1(V W)+ 1(W Y)+ 1 (Y A)

= 1(S A) = b.....2.

Adding 1 and 2, we get

a + b = 1(S R)+1(R Q) + 1(Q P) + 1(P O) + 1(O N)+ 1(N M) + 1(M L) + 1 (L K) + 1(K J)+ 1(J I) + L (I H) + 1 (H G)

= Length of the throne carpet.

This was the calculation which helped Rafi in finding the length of the needed carpet.

Now let us look at this problem from a slightly different angle and see if we stumble across something paradoxical. Let us look at Fig.4.

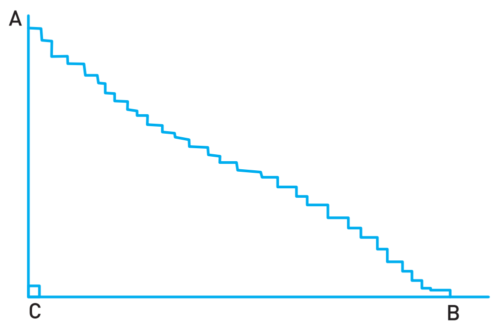

Fig. 4

If we keep on constantly reducing the widths and heights of each step then Fig. 3 gets changed to Fig. 4. AB is thus reduced to a zigzag line. Still the length of this zigzag line is equal to (a + b). Also, we know that as we keep on reducing the size of steps the zigzag line AB resembles more and more a straight line segment AB. In other words, when we reduce the size of each step infinitely, the zigzag line AB becomes a straight line segment. Then our rule tells us that still the length of straight line segment AB should be a + b. But this only means that ΔABC, the sum of the lengths of its two sides is equal to the length of its third side and as we know this is impossible. Then where have we gone wrong to reach this paradox?

Well, the paradox hinges on the fact that we talk of infinity or endless but we always stop the sequence somewhere. We confuse here between infinite number of times and very large number of times. In the above example, one should remember that the length of the zigzag line AB will always be (a + b) no matter how large (but still finite) the number of steps are. But as soon as the number of steps becomes infinite, the rule does not hold and hence the straight line segment AB has a length which is not equal to a + b (but in fact less than a + b).

Catching a Thief

Often, in everyday life, we get indirect information about a variety of things. For example, if a thief steals something and escapes, then police may not learn his name and address directly. Instead, they may get many other small bits of information about the thief. For instance, that he was a tall, fair looking person, or that he was a strong and stout man, etc. Each piece of information thus received brings the police closer and closer to the thief.

This is also true of Mathematics. We often encounter problems where one has to catch the answer hidden in the various pieces of indirect information. For example, instead of telling you directly the number of oranges there are in a store-room, you are told that in the heap 1000 oranges are not good, that 50% of the remaining have been sold and that leaves 3000 oranges. Then one has to find the total number of oranges.

We solve such problems either as a part of our daily work or sometimes we think of them as puzzles for entertainment. Someone not conversant with formal mathematics uses a trial and error method for solving such problems. But a mathematician employs a powerful tool in the solution of such problems. This tool is called an equation. You may wonder what this equation is like. Well, these equations surprisingly resemble our political parties in many respects. Any political party usually pretends to know the answer of peoples' problems. Similarly an equation begins its task by pretending to know the final answer, saying it is X. Now, a political party has its own election symbol such as a lotus or an elephant. So also, an equation has its own symbol which is = (two horizontal line segments). Finally, a political party usually says that it treats rich and poor, literate and illiterate, etc., equally or in other words it is impartial. So also, an equation proceeds on this principle of impartiality, i.e., it always gives identical treatment to its left and right side. i.e., if you divide its right side by 5 then you will have to divide its left side also by 5. If we add 7 to the right side of an equation then we will have to add 7 to its left side too. Using, this principle one can easily solve an equation such as $3x+12=15$. While solving a quadratic equation such as $x^2 + 5x + 6 = 0$, one essentially tries to reduce it to two simple equations $x + 3 = 0$ and $x + 2 = 0$. These equations now can be solved by using the impartiality principle. Even in the solution of simultaneous equations such as $x + y = 5$, $x - 4 = 3$, one tries to reduce them to two simple equations $2x = 8$ and $2y = 2$ which one can solve by using the impartiality principle.

With this much information about the working of equations, let us now turn to a few interesting puzzles recorded in the history of Mathematics. Greek Anthology, Leonardo of Pisa's Liber Abaci and Paciolis Summa give such problems. We will consider here two problems from Anthology and one problem related to the life story of Diophantus who is called the father of Algebra.

(1) Euclid's riddle: A riddle attributed to Euclid and constained in the Anthology is as follows:

Fig.5

A mule and a donkey were walking along laden with corn. The mule says to the donkey, if you gave me one measure I should carry twice as much as you. If I gave you one, we should both carry equal burdens. Tell me their burdens, most learned master of geometry.

Solution: Let the burden carried by the mule be x, and let the burden carried by the donkey be y. Then from the given condition

$(x+1) = 2(y-1)$ ...(1)

and $(x-1) = (y+1)$ ...(2)

Starting with equation (1)

$x + 1 = 2y - 2$

Take out 1 from both sides.

$\therefore x = 2y - 3$ ...(3)

Adding 1 to both sides of equation (2)

We get $x = y + 2$ ...(4)

From equations (3) and (4) we can say that

$2y - 3 = y + 2$ .... (each being equal to x)

Now adding 3 to both sides we get

$2y = y + 5$

Taking out y from both sides gives

$y = 5$

But from equation (2) $x - 1 = y + 1$

i.e. $x - 1 = 5 + 1$

or $x - 1 = 6$

or $x = 7$

Thus the burden of the mule was 7 units and the burden of the donkey was 5 units.

Tally: $(7 - 1) = 5 + 1$

and $(7 + 1) = 2(5 - 1)$

(2) Diophantus' riddle: In the twilight of the Greek era, Diophantus appeared. Though we do not know the exact period in which he lived, we do know how long he lived. We have this information because one of his admirers described his life in the form of an algebraic riddle. It says.

1. Diophantus' youth lasted 1/6 of his life.

2. He grew a beard after 1/12 more.
3. After 1/7 more of his life Diophantus married.
4. Five year later he had a son.
5. The son lived exactly ½ as long as his father and Diophantus died just four years after his son. How long did Diophantus live?

Solution: Let us suppose that he lived for x years.

Hence

(i) His youth was $\frac{x}{6}$

(ii) He grew a beard when $\frac{x}{6}+\frac{x}{12}$ years old.

(iii) He married when he was $\frac{x}{6}+\frac{x}{12}+\frac{x}{7}$ years old.

(iv) He had a son when he was $\frac{x}{6}+\frac{x}{12}+\frac{x}{7}+5$ years old.

(v) His son lived $\frac{x}{2}$ years. Thus when his son died Diophantus was $\frac{x}{6}+\frac{x}{12}+\frac{x}{7}+5+\frac{x}{2}$ years old.

(vi) He died 4 years after his son.

Hence when he died he was

$\frac{x}{6}+\frac{x}{12}+\frac{x}{7}+5+\frac{x}{2}+4$ years old.

(vii) But he lived for x years.

$$\therefore \left(\frac{x}{6}+\frac{x}{12}+\frac{x}{7}+5+\frac{x}{2}+4\right)=x$$

i.e. $5+4=x-\frac{x}{6}-\frac{x}{12}-\frac{x}{7}-\frac{x}{2}$

or $9=\frac{(84\text{-}14\text{-}7\text{-}12\text{-}42)\,x}{84}$

or $9=\frac{9x}{84}$

Multiplying both sides by $\frac{84}{9}$ we get

$\frac{(9)(84)}{9}=x$

or $x = 84$.

Thus Diophantus must have lived for 84 years. Diophantus is called the Father of Algebra because he was the first to abbreviate expression of thoughts with symbols of his own and also because he could solve Indeterminate equations. That is why such equations are often called Diophantine equations.