DANGER OF IMPROPER USE OF MATHEMATICAL RESULTS, FORMULAS AND SYMBOLS : INTEGRALS (CALCULUS)

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The theory of integration is well-known to all mathematicians. Students are also familiar with different results of integration. But this topic is also a subject of serious study like other topics in calculus, which have been explained in a series of articles by the author (1, 2). Students use different standard results of integration in solving a problem without examining their validity. An application of different techniques in finding the integral of a function sometimes gives different results for which students do not find any explanation. Like the previous articles by the same author (1, 2, 3, 4, 5), it will be shown here, with a series of examples, how different results of integration can be obtained due to the lack of knowledge about a theorem and validity of mathematical formulas, etc.

Wrong Use of the Fundamental Theorem of Integral Calculus

Students are familiar with the Fundamental Theorem of Integral Calculus, which is also known as Newton-Leibnitz formula. This theorem can be stated as follows:

$$\int_a^b f(x)dx = F(b) - F(a),$$

Where F(x) is one of the anti-derivatives (or indefinite integrals of f (x) i.e., $F'(x) = f(x)$ in $a \le x \le b$ and f (x) is a continuous function in $a \le x \le b$. It is noticeable that in order that F' (x) exists in $a \le x \le b$, the function F' (x) must also be continuous in $a \le x \le b$. A discontinuous function used as an anti-derivative will lead to the wrong result. Students are familiar with the theorem, but they are not so serious about the nature of the functions f (x) and F (x). The result is, therefore, dangerous, because a student may get the wrong value of an integral, but cannot find any reason for this wrong result. This will be explained with a few examples.

Example 1

Find a mistake in the following evaluation of the integral:

$$\int_0^{\sqrt{3}} \frac{dx}{1+x^2}$$

We know that

$$\frac{d}{dx}\left(\frac{1}{2}\tan^{-1}\frac{2x}{1-x^2}\right) = \frac{d}{dx}\left(\tan^{-1}x\right)$$

$$= \frac{1}{1+x^2}$$

Hence, by the Fundamental Theorem of Integral Calculus, we have

$$\int_0^{\sqrt{3}} \frac{dx}{1+x^2} = \left[\frac{1}{2}\tan^{-1}\frac{2x}{1-x^2}\right]_0^{\sqrt{3}}$$

$$= \frac{1}{2}\left[\tan^{-1}\left(-\sqrt{3}\right) - \tan^{-1}0\right] = -\frac{\pi}{6}$$

Note that the integrand $\frac{1}{1+x^2}$ is positive everywhere in $0 \le x \le \sqrt{3}$, but the area under the curve $y = \frac{1}{1+x^2}$ bounded by the x-axis, the two ordinates $x = 0$ and $x = \sqrt{3}$ comes out to be negative.

This is physically impossible. What is the mistake then in the evaluation of the integral?

To get an explanation, we should examine whether the Fundamental Theorem of Integral Calculus is applicable for the function.

$$\frac{1}{2}\tan^{-1}\frac{2x}{1-x^2} \text{ in } 0 \le x \le \sqrt{3}$$

We observe that the function $\frac{1}{2}\tan^{-1}\frac{2x}{1-x^2}$ is discontinuous for $x = 1$ which lies in $0 \le x \le \sqrt{3}$.

For, $\underset{x \to 1-}{\text{Lt}} \frac{1}{2}\tan^{-1}\frac{2x}{1-x^2} = \frac{\pi}{4}$ and

$$\underset{x \to 1+}{\text{Lt}} \frac{1}{2}\tan^{-1}\frac{2x}{1-x^2} = -\frac{\pi}{4}$$

We have already stated that in order that the Fundamental Theorem of Integral Calculus can be applied to a function in an interval, the function and one of its anti-derivatives should be continuous in that interval. That is why we got an absurd result in the evaluation of an integral.

The correct value of the integral under consideration is equal to $\int_0^{\sqrt{3}} \frac{dx}{1+x^2} = \left[\tan^{-1}x\right]_0^{\sqrt{3}}$

$$= \tan^{-1}\sqrt{3} - \tan^{-1}0 = \frac{\pi}{3},$$

since here $\tan^{-1}x$ is continuous in $0 \le x \le \sqrt{3}$ and the equality $F'(x) = f(x)$ is fulfilled on the whole interval.

Example 2

Evaluate the integral: $\int_0^{\pi} \frac{dx}{1+2\sin^2 x}$

A student may solve the problem as follows:

Let $I = \int \frac{dx}{1+2\sin^2 x}$

Then $I = \int \frac{dx}{\sin^2 x + \cos^2 x + 2\sin^2 x}$

$$= \int \frac{dx}{\cos^2 x + 3\sin^2 x} = \int \frac{\sec^2 x\, dx}{1+3\tan^2 x}$$

Put $\tan x = z$

$\therefore \sec^2 x dx = dz$

$$\therefore I = \int \frac{dz}{1+3z^2} = \frac{1}{3}\int \frac{dz}{z^2 + \left(\frac{1}{\sqrt{3}}\right)^2}$$

$$= \frac{\sqrt{3}}{3}\tan^{-1}\left(\sqrt{3}\,z\right) + C,$$

where C is the constant of integration.

$$\therefore\ I = \frac{1}{\sqrt{3}}\tan^{-1}\left(\sqrt{3}\tan x\right) + C$$

Hence, one of the anti-derivatives can be taken as

$$\frac{1}{\sqrt{3}}\tan^{-1}\left(\sqrt{3}\tan x\right).$$

Hence, the required integral

$$= \frac{1}{\sqrt{3}}\left[\tan^{-1}\left(\sqrt{3}\tan x\right)\right]_0^{\pi} = 0$$, a result which is physically impossible, since the integrand

$\frac{1}{1+2\sin^2 x}$ is everywhere positive in $0 \le x \le \pi$ and hence the area of the region bounded by the curve y =, x-axis and the two ordinates x = 0 and $x = \pi$ cannot be zero. The student may not find any reason for this mistake in the evaluation of an integral in the above method.

The mistake is due to the fact that the anti-derivative $\frac{1}{\sqrt{3}}\tan^{-1}\left(\sqrt{3}\tan x\right)$ is discontinuous at $x = \frac{\pi}{2}$ which lies in the interval $0 \le x \le \pi$. This is because

$$\underset{x \to \frac{\pi}{2}-}{\mathrm{Lt}}\ \frac{1}{\sqrt{3}}\tan^{-1}\left(\sqrt{3}\tan x\right) = \frac{1}{\sqrt{3}}\cdot\frac{\pi}{2} = \frac{\pi}{2\sqrt{3}}$$

and

$$\underset{x \to \frac{\pi}{2}-}{\mathrm{Lt}}\ \frac{1}{\sqrt{3}}\tan^{-1}\left(\sqrt{3}\tan x\right) = \frac{1}{\sqrt{3}}\left(-\frac{\pi}{2}\right) = \frac{\pi}{2\sqrt{3}}$$

Hence, the Fundamental Theorem of Integral Calculus cannot be applied for this anti-derivative.

The correct value of the integral can be obtained in the following way:

$$\text{Let } I = \int \frac{dx}{1+2\sin^2 x}$$

$$= \int \frac{dx}{\cos^2 x + 3\sin^2 x}$$

$$= \int \frac{\operatorname{cosec}^2 x\,dx}{3+\cot^2 x}$$

Put cot x = z.

$\therefore\ \operatorname{cosec}^2 x\,dx = -dz.$

$$\text{Hence, } I = -\int \frac{dz}{3+z^2}$$

$$= -\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{z}{\sqrt{3}}\right) + C$$

$$= -\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{\cot x}{\sqrt{3}}\right) + C.$$

Hence, one of the derivatives is $-\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{\cot x}{\sqrt{3}}\right)$.

This function is continuous in $0 \le x \le \pi$. This is because

$$\underset{x \to 0+}{\mathrm{Lt}}\left[-\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{\cot x}{\sqrt{3}}\right)\right] = -\frac{1}{\sqrt{3}}\tan^{-1}\infty = -\frac{\pi}{2\sqrt{3}}.$$

Also, the value of the function at x = 0 is

$$-\frac{1}{\sqrt{3}}\tan^{-1}\left|\left(\frac{\cot x}{\sqrt{3}}\right)\right|_{x=0} = -\frac{\pi}{2\sqrt{3}}$$

Again,

$$\mathrm{Lt}\left[-\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{\cot x}{\sqrt{3}}\right)\right] = -\frac{1}{\sqrt{3}}\tan^{-1}(-\infty) = \frac{\pi}{2\sqrt{3}}.$$

Also, the value of the function at $x = \pi$ is

$$-\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{\cot x}{\sqrt{3}}\right)\Bigg|_{x=\pi} = \frac{\pi}{2\sqrt{3}}.$$

Further, $-\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{\cot x}{\sqrt{3}}\right)$ is obviously continuous for all x in $0 < x < \pi$. Hence, according to the definition of the continuity of a function in a closed interval, the anti-derivative

$-\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{\cot x}{\sqrt{3}}\right)$ is continuous in $0 \le x \le \pi$.

Thus, the given integral

$$= -\frac{1}{\sqrt{3}}\left[\tan^{-1}\left(\frac{\cot x}{\sqrt{3}}\right)\right]_0^{\pi}$$

$$= -\frac{1}{\sqrt{3}}\left[\tan^{-1}(-\infty) - \tan^{-1}(\infty)\right]$$

$$= -\frac{1}{\sqrt{3}}\left(-\frac{\pi}{2} - \frac{\pi}{2}\right) = \frac{\pi}{\sqrt{3}}$$ which is the correct value of the integral.

It can be mentioned here that the above integral can also be found with the help of the previous function:

$$F(x) = \frac{1}{\sqrt{3}}\tan^{-1}\left(\sqrt{3}\tan x\right).$$

For this purpose, we should divide the interval of integral $(0, \pi)$ into two sub-intervals: $\left(0, \frac{\pi}{2}\right)$ and $\left(\frac{\pi}{2}, \pi\right)$ take into consideration the limiting values of the function F(x) as $x \to \frac{\pi}{2}-$ and $x \to \frac{\pi}{2}+$. Then the anti-derivative becomes a continuous function on each of the sub-intervals and Fundamental Theorem of Integral Calculus is applicable.

Thus we have

$$\int_0^{\pi} \frac{dx}{\cos^2 x + 3\sin^2 x}$$

$$= \int_0^{\pi/2} \frac{dx}{\cos^2 x + 3\sin^2 x} + \int_{\pi/2}^{\pi} \frac{dx}{\cos^2 x + 3\sin^2 x}$$

$$= \frac{1}{\sqrt{3}}\left[\tan^{-1}\left(\sqrt{3}\tan x\right)\right]_0^{\pi/2} + \frac{1}{\sqrt{3}}\left[\tan^{-1}\left(\sqrt{3}\tan x\right)\right]_{\pi/2}^{\pi}$$

$$= \frac{1}{\sqrt{3}}\left[\frac{\pi}{2} - 0 + 0 - \left(-\frac{\pi}{2}\right)\right] = \frac{\pi}{\sqrt{3}}.$$

Error due to the Neglect of the Sign of an Expression under a Square Root

Sometimes we ignore the sign of an expression under the square root in an integral throughout the limits of integration and as a result, we come across with a wrong answer. We give an example to explain this.

Example 3

Evaluate: $\int_0^{\pi} \sqrt{\frac{1+\cos 2x}{2}}dx$.

Most of the students will perhaps evaluate this integral in the following manner:

$$\int_0^{\pi/2} \sqrt{\frac{1+\cos 2x}{2}}dx = \int_0^{\pi} \sqrt{\frac{2\cos^2 x}{2}}dx$$

$$= \int_0^{\pi} \cos x dx = \left[\sin x\right]_0^{\pi} = 0.$$

But the answer is wrong. This is due to the fact that they have not considered two different signs of cos x in $(0, \pi)$. It is well-known that $\cos x \ge 0$ when $0 \le x \le \frac{\pi}{2}$ and $\cos x \le 0$ when $\frac{\pi}{2} \le x \le \pi$. Hence,

$$\sqrt{\cos^2 x} = \cos x \text{ if } 0 \le x \le \frac{\pi}{2}$$

$= -\cos x$ if $\frac{\pi}{2} \le x \le \pi$.

In the above solution, students have taken $\sqrt{\cos^2 x} = \cos x$ throughout the interval $[0, \pi]$ and hence they got the incorrect answer. The correct solution will be as follows:

$$\int_0^{\pi} \sqrt{\frac{1+\cos 2x}{2}} dx$$

$$= \int_0^{\pi} \sqrt{\cos^2 x} dx$$

$$= \int_0^{\pi/2} \cos x dx + \int_{\pi/2}^{\pi} (-\cos x) dx$$

$$= [\sin x]_0^{\pi/2} - [\sin x]_{\pi/2}^{\pi} = 1 + 1 = 2.$$

With similar arguments, we can show that

$$\int_0^{100\pi} \sqrt{1 - \cos^2 x} dx = \int_0^{100\pi} \sqrt{2\sin^2 x} dx$$

$$= \sqrt{2} \int_0^{100\pi} |\sin x| dx = 200\sqrt{2}.$$

Error due to Making No Distinction between Proper and Improper Integrals

Sometimes students do not make any difference between a proper integral and an improper integral and they calculate the improper integral in the same way as they calculate a proper integral. This is sometimes very dangerous because they are likely to get a wrong answer. To explain this, we give an example:

Example 4

Evaluate: $\int_{-1}^{1} \frac{dx}{x^2}$

Without examining whether it is a proper integral or an improper integral, students generally evaluate the integral in the usual way, as discussed below:

$$\int_{-1}^{1} \frac{dx}{x^2} = \left[-\frac{1}{x}\right]_{-1}^{1} = -1 - 1 = -2.$$

But this answer is wrong, because it will be shown that the integral does not exist. The mistake is due to the fact that the integrand $\frac{1}{x^2}$ is discontinuous at $x = 0$ which lies in $[-1, 1]$. Hence the Fundamental Theorem of Integral Calculus cannot be applied here. The integral is, in fact, an improper integral. The correct solution of the problem is as follows:

$$\int_{-1}^{1} \frac{dx}{x^2} = \int_{-1}^{0} \frac{dx}{x^2} + \int_{0}^{1} \frac{dx}{x^2}.$$

$$= \underset{\epsilon_1 \to 0+}{\text{Lt}} \int_{-1}^{-\epsilon_1} \frac{dx}{x^2} + \underset{\epsilon_2 \to 0+}{\text{Lt}} \int_{\epsilon_2}^{1} \frac{dx}{x^2}$$

The function $\frac{1}{x^2}$ is obviously continuous in $-1 \le x \le -\epsilon_1$ as well as in $\epsilon_2 \le x \le 1$ for $\epsilon_1, \epsilon_2 > 0$. Hence, applying the Fundamental Theorem of Integral Calculus, we get

$$\int_{-1}^{-\epsilon_1} \frac{dx}{x^2} = -\left[\frac{1}{x}\right]_{-1}^{-\epsilon_1} = \frac{1}{\epsilon_1} - 1$$

and

$$\int_{\epsilon_2}^{1} \frac{dx}{x^2} = -\left[\frac{1}{x}\right]_{\epsilon_2}^{1} = -1 + \frac{1}{\epsilon_2}$$

Now, since $\underset{\epsilon_1 \to 0+}{\text{Lt}} \frac{1}{\epsilon_1}$ and $\underset{\epsilon_2 \to 0+}{\text{Lt}} \frac{1}{\epsilon_2}$ do not exist,

Hence the given integral does not exist.

Making the Proper Integral Improper by Some Operation

In order to simplify an integrand, students sometimes make some operations with its numerator and denominator without thinking about the nature of the integrands in the two cases. As a result, they sometimes obtain an absurd result for which they do not get any explanation. We consider an example to explain this type of error.

Example 5

Evaluate: $\int_0^{\pi/2} \frac{dx}{1+\cos x}$

Many students try to evaluate the integral in the following manner:

$$\int_0^{\pi/2} \frac{dx}{1+\cos x} = \int_0^{\pi/2} \frac{1-\cos x}{1-\cos^2 x}dx$$

[Multiplying the numerator and denominator of the integrand by 1-cos x]

$$= \int_0^{\pi/2} \frac{1-\cos x}{\sin^2 x}dx$$

$$= \int_0^{\pi/2} \operatorname{cosec}^2 x\ dx - \int_0^{\pi/2} \cot x\ \operatorname{cosec}\ x\ dx$$

$$= -\left[\cot x\right]_0^{\pi/2} + \left[\operatorname{cosec}\ x\right]_0^{\pi/2}$$

Which is undefined, since cot 0 and cosec 0 are both infinity.

It is noticeable that the original integrand $\frac{1}{1+\cos x}$ is continuous for all x in $0 \le x \le \frac{\pi}{2}$ and hence the integral should exist. The reason for this contradiction is that multiplying numerator and the denominator of the integrand by 1 – cos x which is zero for x = 0, we have made the original integrand discontinuous. In other words, we have changed the original proper integral to an improper integral: $\int_0^{\pi/2} \frac{1-\cos x}{1-\cos^2 x}dx$. But these two integrals are completely different. Since a continuous function can never be equated with a discontinuous function by any operation, multiplication of the numerator and the denominator of the original integrand by 1 – cos x in this particular example is not valid. The correct method of solution is, therefore, as follows:

$$\text{Given integral} = \int_0^{\pi/2} \frac{dx}{\cos^2 \frac{x}{2}}$$

$$= \frac{1}{2}\int_0^{\pi/2} \sec^2 \frac{x}{2} dx$$

$$= \left[\tan \frac{x}{2}\right]_0^{\pi/2} = 1.$$

Improper Substitution in a Definite Integral

No student perhaps thinks much about the validity of a substitution in a definite integral. He becomes satisfied whenever a substitution changes an integral to a standard integral. The result is sometimes dangerous. Consider the following example:

Example 6

Evaluate: $\int_{-1}^{1} \frac{dx}{1+|x|}$

A student may substitute $x = t^2$ whence $dx = 2tdt$ in order to remove the modulus sign from the integrand. As a result, we observe that when $t = -1$, $x = 1$ and when $t = 1$, $x = 1$ so that the integral becomes $\int_{1}^{1} \frac{2\,tdt}{1+t^2} = 0$, since the upper and the lower limits of the integral are equal. But this result is wrong, since the integrand is greater than zero for all x in $-1 \le x \le 1$ and so we expect a non-zero positive value for the integral. The mistake is due to the fact that the substitution $x = t^2$ is not valid in this case. This is because the substitution $x = t^2$ makes x always positive whereas in the given interval x may be positive or negative in $-1 \le x \le 1$. We can easily evaluate the integral in the following manner:

$$\text{Given integral} = \int_{-1}^{0} \frac{dx}{1+|x|} + \int_{0}^{1} \frac{dx}{1+|x|}$$

$$= -\int_{-1}^{0} \frac{dx}{x-1} + \int_{0}^{1} \frac{dx}{1+x}$$

$$= -\left[\log|x-1|\right]_{-1}^{0} + \left[\log|1+x|\right]_{0}^{1}$$

$$= = \log 2 + \log 2 = 2\log 2.$$

Conclusion

The series of examples given in the series of articles by the author (1, 2) from calculus on limits, derivatives and integrals are only a few of many such examples. Readers themselves may find a lot of such fallacious results in calculus while solving problems. Similar fallacious results in the topics of algebra, coordinate geometry and trigonometry were shown previously by the author (3, 4, 5). Whenever such fallacious results appear in any branch of mathematics, one should recall and think of the possible restrictions involved in a formula, result, theorem, etc., which have been used to solve a problem. The examples cited in the articles will perhaps convince any reader that studies of any theorem, formula, definition, symbol in mathematics need very serious attention. The purpose of the series of such papers under the same heading on different topics of mathematics, which are at present taught in different classes from Classes IX to XII is to make the students and teachers alert about the danger of improper use of mathematical results, formulas, etc. The author will consider his labour to be fruitful even if a few teachers and students of our country are benefitted by the perusal of these articles.

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