DANGER OF IMPROPER USE OF MATHEMATICAL RESULTS, FORMULAS AND SYMBOLS : LIMIT (CALCULUS)

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A lot of work has been done for the improvement of mathematics teaching in schools. Many textbooks in mathematics are also available in our country, where various mathematical problems have been worked out. But the study of the basic theory of mathematics involving theorems, formulas, symbols, etc., has not been so emphasised. Its consequence is very dangerous. It has been shown by the author [1, 2] with examples how fallacious results may be obtained while solving problems of algebra and trigonometry due to the lack of the basic concept in the theory. With the introduction of Calculus in the schools, it is necessary to point out similar fallacious results in Calculus, which may be obtained due to the improper use of these formulas, theorems, etc. In the present article which deals with the topic 'Limit', it will be shown with examples how the majority of students, while solving a problem, apply theorems and formulas of limit wrongly without considering the applicability of these results. It can be mentioned here that many textbooks in mathematics available in the market now-a-days also contain this type of conceptual errors. One of the purposes of this article is, therefore, to make the teachers and students aware of these errors.

Errors due to the Limit of the Sum of Functions

In finding the limit of a function, we make use of the fundamental theorems of limit without examining whether the theorems are applicable or not.

One of the fundamental theorems is that the limit of the sum of a finite number of functions is equal to the sum of the limits of the functions. In finding the limit of the sum of a number of functions, we generally do not bother about the number of functions and apply the above theorem. As a result, we sometimes get wrong answers. We consider some examples:

Example 1: Find the following limit:

$$\underset{n\to\infty}{Lt}\left(\frac{1}{n}+\frac{1}{n}+\frac{1}{n}+ \text{.....to n terms}\right).$$

Many students will find the limit by applying the theorem stated above. If we apply this theorem, then we shall get:

Given Limit

$$= \underset{n\to\infty}{Lt}\frac{1}{n}+\underset{n\to\infty}{Lt}\frac{1}{n}+\underset{n\to\infty}{Lt}\frac{1}{n}+ \text{...... to n terms}$$

= 0 + 0 + 0 + ... to n terms

= 0.

But unfortunately the limiting value is not 0. It is 1. So, the above method of solution is wrong. The correct solution is as follows:

Given Limit

$= \underset{n\to\infty}{Lt}\left(\frac{1}{n}\times n\right)$, (adding $\frac{1}{n}$, n times)

$= \underset{n\to\infty}{Lt}(1) = 1$.

Now, the question is: "Why is the previous method wrong?" This is because the number of functions is n and as n $\to \infty$ the number of functions also becomes infinite. The sum formula of the limit is applicable only when the number of functions is finite. In the above problem, since the number of functions is not finite, hence the above theorem cannot be applied here in solving the problem.

Example 2: We now consider the following limit:

$$\underset{n\to\infty}{Lt}\left[\frac{1}{n^2}+\frac{2}{n^2}+\frac{3}{n^2}+.....+\frac{n-1}{n^2}\right]$$

Using the sum formula, we get the given limit as

$$\underset{n\to\infty}{Lt}\frac{1}{n^2}+\underset{n\to\infty}{Lt}\frac{2}{n^2}+\underset{n\to\infty}{Lt}\frac{3}{n^2}+.....+\underset{n\to\infty}{Lt}\frac{n-1}{n^2}$$

= 0 + 0 + 0 + ... + 0, (n zeros)

= 0.

Alternatively, if we add the n-1 functions first and then take the limit, we get the given limit as

$$\underset{n\to\infty}{Lt}\frac{(n-1)n}{2n^2}+\underset{n\to\infty}{Lt}\frac{n^2-n}{2n^2}-\underset{n\to\infty}{Lt}\left(\frac{1}{2}-\frac{1}{2n}\right)$$

$$=\frac{1}{2}-0=\frac{1}{2}.$$

The second answer, i.e., $\frac{1}{2}$, is the correct answer and the former answer is wrong, since we get this answer by the wrong use of the sum formula which is applicable in the case of a finite number of terms. The number of terms of the given function is n-1 and as $n\to\infty$, the number of terms becomes infinite. Hence, the sum formula cannot be applied in solving the problem.

We now consider the limit of an infinite series and show that taking the limit term by term of an infinite series leads one to get wrong answer. We explain it with the help of an example.

Example 3: Consider the series:

$x^2+\frac{x^2}{1+x^2}+\frac{x^2}{\left(1+x^2\right)^2}+......\text{to }\infty$, and suppose that we are interested to find the limit of this series as $x\to 0$.

If we take the limit term by term, i.e., if we apply the sum formula, then we get

$$\underset{x\to\infty}{Lt}\left[x^2+\frac{x^2}{1+x^2}+\frac{x^2}{\left(1+x^2\right)^2}+......\text{to }\infty\right]$$

$$=\underset{x\to\infty}{Lt}x^2+\underset{x\to\infty}{Lt}\frac{x^2}{1+x^2}+\underset{x\to\infty}{Lt}\frac{x^2}{\left(1+x^2\right)^2}+......\text{to }\infty$$

= 0 + 0 + 0 + ...to ∞

=0.

But the above answer is wrong due to the fact that we have violated the restriction on the number of terms of an infinite series. The correct solution is, therefore, as follows:

The given series is an infinite G.P. series with common ratio $\frac{1}{1+x^2}$. Hence, the sum of the G.P. series is

$$x^2 \frac{1}{1-\frac{1}{1+x^2}} = 1 + x^2$$

Hence,

$$\underset{x\to 0}{Lt}\left[x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{\left(1+x^2\right)^2} + \ldots\ldots \text{to } \infty\right]$$

$$= \underset{x\to 0}{Lt}\left(1+x^2\right) = 1.$$

To explain why term by term limit of an infinite series is not allowed, we first state the meaning of the sum of an infinite series.

We consider the following infinite series:

$f_1(x)+f_2(x) + f_3(x) +\ldots\ldots\ldots+ f_n(x) +\ldots\ldots\ldots$to ∞.

Let Sn(x) = $f_1(x) + f_2(x) + \ldots\ldots\ldots\ldots\ldots\ldots + f_n(x)$.

Sn(x), as is well-known, is called the partial sum of the infinite series.

Let $\underset{n\to\infty}{Lt}$ Sn(x) be a finite number, say $n \to \infty$ S. Then S will be called the sum of the series. Hence, the sum of an infinite series is, in fact, the limit of the partial sum Sn(x) as $n \to \infty$ provided this limit is finite.

Now, to explain why we cannot proceed to the limit term by term, we take the following problem involving two limits:

Consider the expression $\frac{x-y}{x+y}$ and suppose that we want to find its limit as x and y both tend to 0. If we let x $\to$ 0 first and then tend y to zero, we shall have

$$\underset{y\to 0}{Lt}\left[\underset{x\to 0}{Lt}\frac{x-y}{x+y}\right] = \underset{y\to 0}{Lt}\left(\frac{-y}{y}\right) = \underset{y\to 0}{Lt}\,(-1) = -1.$$

If, on the other hand, we let y $\to$ 0 first and then tend x to zero, we shall have

$$\underset{x\to 0}{Lt}\left[\underset{y\to 0}{Lt}\frac{x-y}{x+y}\right] = \underset{x\to 0}{Lt}\frac{x}{x} = \underset{x\to 0}{Lt}\,(1) = 1.$$

Thus we get two different results.

In the light of the above example, the problem of the limit of the infinite series can be explained.

We have S(x) = $\underset{n\to\infty}{Lt}$ Sn(x).

We now take the limit of the sum function as x $\to$ a, say.

Hence, $\underset{n\to\infty}{Lt}\ S(x) = \underset{x\to a}{Lt}\left[\underset{x\to\infty}{Lt}\ S(x)\right]$, (1)

while the series obtained by taking the limit term by term is

$$\underset{x\to a}{Lt}\ f_1(x) + \underset{x\to a}{Lt}\ f_2(x) + \ldots\ldots\text{.to } \infty$$

$$= \underset{n\to\infty}{Lt}\left[\underset{x\to a}{Lt}\ f_1(x) + \underset{x\to a}{Lt}\ f_2(x) + \ldots\ldots + \underset{x\to a}{Lt}\ f_n(x)\right]$$

$$= \underset{n\to\infty}{Lt}\left[\underset{x\to a}{Lt}\ Sn(x)\right] \qquad (2)$$

There is no reason to believe that the double limit shown in (1) and (2) will be equal.

There is, however, a special type of infinite series known as power series in which term by term differentiation is possible within the interval of convergence. The discussion of the convergence of this type of series is beyond the scope of the school syllabus. School students should, therefore, be advised not to take the limit of an infinite series term by term in order to avoid possible mistakes.

Errors Due to the Limit of the Product of Functions

Another fundamental theorem of limit is that the limit of the product of two functions is the product of their limits, provided each limit exists. Students generally do not emphasise the restriction involved in the theorem, i.e., existence of each limit.

For example, consider the following limit:

$$\underset{x\to 0}{Lt}\left(x\sin\frac{1}{x}\right)$$

In finding this limit, many students proceed as follows:

$$\underset{x\to 0}{Lt}\left(x\sin\frac{1}{x}\right)$$

$$= \underset{x\to 0}{Lt}\ x \times \underset{x\to 0}{Lt}\ \sin\frac{1}{x} = 0 \times \underset{x\to 0}{Lt}\ \sin\frac{1}{x} = 0.$$

The limiting value is accidentally correct, but the method of solution is wrong due to the fact that $\underset{x\to 0}{Lt}\ x\sin\frac{1}{x}$ does not exist. The theorem on the limit of the product of two functions is applicable only if the limit of each function exists.

The correct method of solution of the above problem can be obtained by giving argument that whatever small value x may have, $\sin\frac{1}{x}$ is always finite (although it is not known definitely) and it lies in the closed interval (-1, 1). Since, $\underset{x\to 0}{Lt}\ x = 0$ hence the limit of x multiplied by a finite number will also be 0 as $x\to 0$.

Students may be curious to know why this restriction is necessary while they get the correct answer even after the violation of the restriction of the theorem. That they may not always get the correct answer by violating this restriction can be shown with an example:

Example 4: Find the value of

$$\underset{x\to a+}{Lt}\ \frac{1}{x-a}\times\left(x^2-a^2\right).$$

If we apply the theorem on the limit of the product of two functions, then we get the given limit as

$$\underset{x\to a+}{Lt}\ \frac{1}{x-a}\times \underset{x\to a+}{Lt}\left(x^2-a^2\right)$$

$= \infty \times 0$ which is indeterminate. Hence, the limit does not exist. But this conclusion is wrong and it will be shown now that this limit exists and hence is finite. The correct answer is as follows:

$$\underset{x\to a^+}{Lt}\ \frac{1}{x-a}\times\left(x^2-a^2\right).$$

$$= \underset{x\to a+}{Lt}\ \frac{(x-a)(x+a)}{x-a}$$

$$= \underset{x\to a+}{Lt}\ (x+a) = 2a$$

which is the correct answer.

The previous method of solution is wrong because $\underset{x\to a+}{Lt}\ \frac{1}{x-a}$ does not exist. We cannot, therefore, use the theorem of the limit of the product of two functions. The above example is one of many to show that students are likely to get incorrect answers due to the violation of the restriction on the above theorems.

Errors Due to Negligence of Symbols

The negligence of certain symbols used in the theory of limit or the lack of knowledge in it leads students to get incorrect answers. Students generally give hardly any importance to the notations like $\underset{x \to a+}{Lt} f(x)$ and $\underset{x \to a-}{Lt} f(x)$.

They sometimes make no distinction between these two concepts although they may be familiar with the symbols representing the right-hand limit and the left-hand limit. As a result, they are likely to get wrong answers. Some examples are given below to explain it.

Example 5: Evaluate , $\underset{x \to 1}{Lt} \sqrt{1-x}$ if possible.

Many students will say that the above limit exists and the limiting value is 0, since $1-x \to 0$ as $x \to 1$. But this conclusion is wrong due to the fact that if $x > 1$, then $\sqrt{1-x}$ is imaginary and hence $\underset{x \to 1+}{Lt} \sqrt{1-x}$ does not exist although $\underset{x \to 1-}{Lt} \sqrt{1-x}$ exists and is equal to 0.

The correct answer to the given problem is that the limit does not exist. The mistake done by the students is due to the fact that without examining $\underset{x \to 1+}{Lt} \sqrt{1-x}$ and $\underset{x \to 1-}{Lt} \sqrt{1-x}$. separately, they simply put $x = 1$ in $\sqrt{1-x}$ and get a wrong answer.

We consider below another interesting problem where students are likely to do the above type of mistakes due to the negligence of the symbols.

Example 6: It is a well-known result of Coordinate Geometry that the product of the slopes of two perpendicular lines is always -1. Explain whether the result is true for x and y axes which are perpendicular to each other.

A student may analyse the problem like this: The slope of the x-axis is 0 and that of y-axis is ∞. Hence, their product is $\infty \times 0$ which is indeterminate. Hence, the result is not true for x and y axes. But this conclusion of the student is wrong, since it is a well-known theorem of Coordinate Geometry that the result is true for any pair of perpendicular lines.

A more intelligent student may analyse the problem by the process of limit. He may argue as follows: We denote the x and y axes by Ox and Oy (Fig. 1) and let Ox′ and Oy′ make small angle θ with x and y axes as shown. We shall find the product of the slopes of Ox′ and Oy′ in the limit when $\theta \to 0$ in which case Ox′ and Oy′ will coincide with x and y axes respectively.

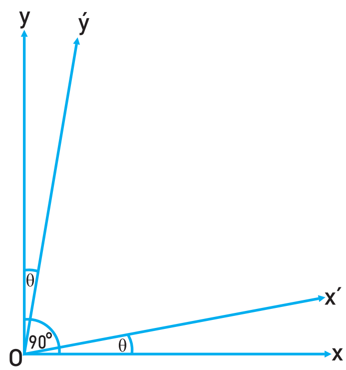

Fig. 1

Now, the slope of Ox′ = tan θ and the slope of Oy′ is $\tan(90^0 - \theta) = \cot\theta$. The limit of the product of these slopes as $\theta \to 0$ is $\underset{\theta \to 0}{Lt} \tan\theta \cot\theta = \underset{\theta \to 0}{Lt} (1) = 1,$ which also shows that the result is not true, since the product of the slopes in the limit should be -1.

The above wrong answer is due to the fact that we have not considered whether $\theta \to 0+$ or, $\theta \to 0-$. If, $\theta \to 0+$, then Ox′ moves in the clockwise sense and in order that Oy′ tends to coincide with Oy, the rotation of Oy′ should be anticlockwise and hence $\theta \to 0-$ for Oy′ to coincide with Oy. It is, therefore, evident that while we write $\theta \to 0$ to find the limit of the product of the slopes of Ox' and Oy' we are actually considering $\theta \to 0+$ as well as $\theta \to 0-$ simultaneously, which is absurd.

The figure should, therefore, be drawn in such a manner that Ox' and Oy' move in the same direction keeping the angle between them as $\frac{\pi}{2}$ and coincide with Ox and Oy in the limit. This figure is shown below:

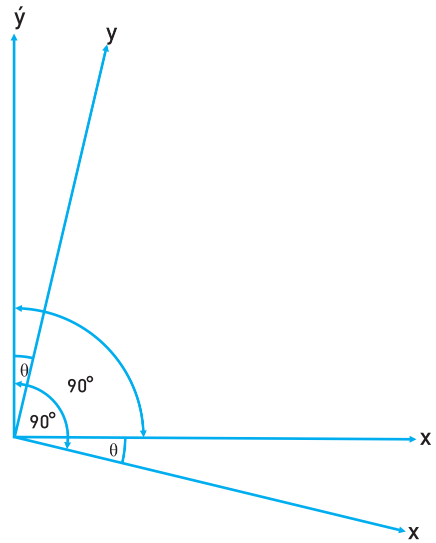

Fig. 2

According to Fig. 2, the slope of Ox' tan θ and that of Oy' = tan $(90^0 + \theta) = -\cot\theta$.

Hence, the limit of the product of the slopes of the two lines is $\underset{\theta \to 0+}{Lt} \tan\theta\,(-\cot\theta) = \underset{\theta \to 0+}{Lt}(-1) = -1$. (In order that Ox' and Oy' coincide with Ox and Oy respectively, θ should tend to 0 + and not 0–). Hence, the above result is true even for x and y axes also. The above examples will show how dangerous it is to overlook the importance of some symbols in mathematics.

References

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